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On Homogeneous and Symmetric *CR* Manifolds

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Dedicated to the memory of Professor Aldo Andreotti
on the 30th anniversary of his death.

Abstract. – *We consider canonical fibrations and algebraic geometric structures on homogeneous CR manifolds, in connection with the notion of CR algebra. We give applications to the classifications of left invariant CR structures on semisimple Lie groups and of CR-symmetric structures on complete flag varieties.*

Introduction.

In this paper we discuss some topics about the *CR* geometry of homogeneous manifolds. Our main tool are *CR* algebras, introduced in [28] to parametrize homogeneous partial complex structures. If M is a $\boldsymbol{G}_0$-homogeneous *CR*-manifold, we associate to each point p_0 of M a pair $(\mathfrak{g}_0, \mathfrak{q})$, consisting of the Lie algebra $\mathfrak{g}_0$ of $\boldsymbol{G}_0$ and of a complex Lie subalgebra of its complexification $\mathfrak{g}$. If $p = g{\cdot}p_0$, with $g \in \boldsymbol{G}_0$, is another point of M, the *CR* algebra of M at p is $(\mathfrak{g}_0, \mathrm{Ad}(g)(\mathfrak{q}))$, so that $\mathfrak{q}$ is determined by M modulo $\boldsymbol{G}_0$-equivalence. Several questions about the *CR* geometry of M can be conveniently reduced to Lie-algebraic questions about the pair $(\mathfrak{g}_0, \mathfrak{q})$. This program has already been started and carried on in several papers, see e.g. [1, 2, 23, 24], where the investigation focused on different special classes of homogeneous *CR* manifolds. In [20], W. Kaup and D. Zaitsev introduced the notion of *CR*-symmetry, generalizing at the same time the Riemannian and Hermitian cases, and showing that *CR*-symmetric manifolds are *CR*-homogeneous.

In [23, 24] one of the Authors, in collaboration with A. Lotta, classified and investigated some classes of *CR*-symmetric manifolds. A key point was the possibility of representing the partial complex structure of M by an inner derivation J of $\mathfrak{g}_0$. The existence of such a J was a crucial step in the classification of semisimple Levi-Tanaka algebras in [25], and in establishing the structure of standard *CR* manifolds, which are homogeneous *CR* manifolds with maximal *CR*-automorphisms groups (see [26, 28]). In this paper we will delve further into the relationship between the existence of J, canonical *CR* fibrations, and *CR*-symmetry.

In the first section we survey the basic notions of *CR* and homogeneous *CR* manifolds, including the J-property, *CR*-symmetry, and explaining their relationship.

In § 2, we discuss the existence of Levi-Malcev and Jordan-Chevalley fibrations, and the existence of suitable homogeneous *CR* structures on their total spaces, bases and fibers. These fibrations, interwoven with canonical decompositions of the *CR* algebras, were largely employed in [26, 27], and also in [1, 2] in the context of parabolic *CR* manifolds. Here they are considered in full generality.

In § 3 we study the inverse problem of constructing a $\boldsymbol{G}_0$-homogeneous *CR* manifold starting from an abstractly given *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$. This is not always possible, and the question arises to describe natural modifications of $(\mathfrak{g}_0, \mathfrak{q})$ leading to new $\boldsymbol{G}_0$-homogeneous *CR* manifolds. These are described in § 3.2, § 4.4, § 4.5.

The construction of § 4.5 was employed in [4, 15] and, like the one of § 4.4, has a distinct algebraic geometrical flavor. Besides, algebraic groups played a central role in the study of parabolic *CR* manifolds in [1, 2]. Thus we consider *CR* manifolds in an algebraic geometric context in § 4. We show that algebraic *CR* manifolds canonically embed into the set of regular points of complex algebraic varieties. An important distinction arises between algebraic and weakly-algebraic *CR* manifolds, the latter admitting analytic, but not algebraic, embeddings.

In the two final sections we deal with special applications. In § 5 we extend to noncompact Lie groups some results of [9], classifying the regular left invariant maximal *CR* structures on semisimple real Lie groups. In § 6, we consider symmetric *CR* structures on full flags of complex Lie groups. They had been considered in [14] in a slightly different context. In our treatment we use the *CR* algebras approach and we are especially interested in the relationship between *CR*-symmetry and the J-property. All the *CR*-symmetric manifolds of [23, 24] also enjoyed the J-property. We are in the same situation when we consider the complete flags of the classical groups. On the complete flags of the exceptional groups we found examples of *CR*-symmetric structures which do not enjoy even a weaker version of the J-property, and also examples of *CR*-structures enjoying the weak-J-property, but not the J-property.

1. – *CR* manifolds, *CR* algebras, *J*-property, *CR*-symmetry.

1.1 – *CR manifolds.*

Let M be a smooth real manifold. A *CR structure* on M is the datum of an almost Lagrangian formally integrable smooth complex subbundle $T^{0,1}M$ of the complexified tangent bundle $T^{\mathbb{C}}M$. The subbundle $T^{0,1}M$ is required to satisfy:

$$T^{0,1}M \cap \overline{T^{0,1}M} = 0, \tag{1.1}$$

$$[\Gamma(M, T^{0,1}M), \Gamma(M, T^{0,1}M)] \subset \Gamma(M, T^{0,1}M). \tag{1.2}$$

The rank n of $T^{0,1}M$ is the *CR-dimension*, and $k = \dim_{\mathbb{R}} M - 2n$ the *CR-codimension* of M. If $n = 0$, we say that M is totally real; if $k = 0$, M is a complex manifold in view of the Newlander-Nirenberg theorem.

When M is a real submanifold of a complex manifold $\boldsymbol{X}$, for every $p \in M$ we can consider the $\mathbb{C}$-vector space $T^{0,1}_p M = T^{0,1}_p \boldsymbol{X} \cap T^{\mathbb{C}}_p M$ of the anti-holomorphic complex tangent vectors on $\boldsymbol{X}$ which are tangent to M at p. If the dimension of $T^{0,1}_p M$ is independent of $p \in M$, then $T^{0,1}M = \bigcup_{p\in M} T^{0,1}_p M$ is an almost Lagrangian formally integrable complex subbundle of $T^{\mathbb{C}}M$, defining on M the structure of a *CR submanifold* of $\boldsymbol{X}$. If the complex dimension of $\boldsymbol{X}$ is the sum of the *CR*-dimension and the *CR*-codimension of M, the embedding $M \hookrightarrow \boldsymbol{X}$ is called *CR-generic*.

A smooth map $f : M' \to M$ is *CR* if M and M' are *CR* manifolds, and $df(T^{0,1}M') \subset T^{0,1}M$. The notions of *CR* immersion, submersion, diffeomorphism and automorphism are defined in an obvious way. The set of all *CR* automorphisms of a *CR* manifold M is a group that we denote by $\mathrm{Aut}_{CR}(M)$.

DEFINITION 1.1 (Characteristic bundle and Levi forms). – *Let HM be the subbundle of TM consisting of the real parts of the elements of $T^{0,1}M$. Its annihilator bundle $H^0M \subset T^*M$ is called the* characteristic bundle *of M. We have*

$$(1.3)\qquad H^0_pM = \{\xi \in T^*_pM \mid \xi(z) = 0,\ \forall z \in T^{0,1}_pM\}, \quad \text{for all } p \in M.$$

If Z_1, Z_2 are smooth sections of $T^{0,1}M$, and Ξ a smooth section of H^0M, all defined on an open neighborhood of p in M, with $Z_i(p) = z_i$ and $\Xi(p) = \xi$, then we set

$$(1.4)\qquad \mathcal{L}_\xi(z_1, z_2) = i d\Xi(z_1, \bar{z}_2) = -i\xi([Z_1, \bar{Z}_2]).$$

In this way we define a Hermitian symmetric form $\mathcal{L}_\xi$ on $T^{0,1}_pM$, which is called the scalar Levi form *at $\xi \in H^0M$.*

If Z is a smooth section of $T^{0,1}M$ defined on a neighborhood of $p \in M$, with $Z(p) = z$, we define

$$(1.5)\qquad \mathfrak{L}_p(z) = \varpi_p(i[\bar{Z}, Z]_p),$$

where $\varpi_p : T_pM \to T_pM/H_pM$ is the projection into the quotient.

This map $\mathcal{L}_p : T^{0,1}_pM \to T_pM/H_pM$ is the vector valued Levi form of M at p.

1.2 – *Homogeneous CR manifolds.*

Let M be a smooth *CR* manifold and $\boldsymbol{G}_0$ a Lie group.

DEFINITION 1.2. – *We say that M is a $\boldsymbol{G}_0$-homogeneous CR manifold if $\boldsymbol{G}_0$ acts transitively on M by CR diffeomorphisms.*

Let M be a $\boldsymbol{G}_0$-homogeneous CR manifold. Fix $p_0 \in M$, let $\boldsymbol{I}_0 = \{g \in \boldsymbol{G}_0 \mid g \cdot p_0 = p_0\}$ be the isotropy subgroup, and $\pi : \boldsymbol{G}_0 \to M \simeq \boldsymbol{G}_0/\boldsymbol{I}_0$ the associated principal $\boldsymbol{I}_0$-bundle. Denote by $\mathfrak{Z}(\boldsymbol{G}_0)$ the space of smooth sections of the pullback $\pi^* T^{0,1}M$ of $T^{0,1}M$ to $\boldsymbol{G}_0$:

$$\mathfrak{Z}(\boldsymbol{G}_0) = \{Z \in \mathfrak{X}^{\mathbb{C}}(\boldsymbol{G}_0) \mid \pi_* Z_g \in T^{0,1}_{\pi(g)}M, \ \forall g \in \boldsymbol{G}_0\}, \tag{1.6}$$

where $\mathfrak{X}^{\mathbb{C}}(\boldsymbol{G}_0)$ is the space complex valued smooth vector fields on $\boldsymbol{G}_0$. By (1.2), the complex system $\mathfrak{Z}(\boldsymbol{G}_0)$ is formally integrable, i.e.

$$[\mathfrak{Z}(\boldsymbol{G}_0), \mathfrak{Z}(\boldsymbol{G}_0)] \subset \mathfrak{Z}(\boldsymbol{G}_0). \tag{1.7}$$

Moreover, $\mathfrak{Z}(\boldsymbol{G}_0)$ is invariant by left translations, and therefore is generated, as a left $\mathcal{C}^\infty(\boldsymbol{G}_0, \mathbb{C})$-module, by its left invariant vector fields.

Let $\mathfrak{g}_0$ be the Lie algebra of $\boldsymbol{G}_0$ and $\mathfrak{g}$ its complexification. By (1.7), the left invariant elements of $\mathfrak{Z}(\boldsymbol{G}_0)$ define a complex $\mathrm{Ad}_{\mathfrak{g}}(\boldsymbol{I}_0)$-invariant Lie subalgebra $\mathfrak{q}$ of $\mathfrak{g}$, given by

$$\mathfrak{q} = \pi_*^{-1}(T^{0,1}_{p_0}M) \subset \mathfrak{g} \simeq T^{\mathbb{C}}_e \boldsymbol{G}_0. \tag{1.8}$$

We can summarize these observations by

PROPOSITION 1.3. – *Let $\boldsymbol{G}_0$ be a Lie group, $\boldsymbol{I}_0$ a closed subgroup of $\boldsymbol{G}_0$, $\mathfrak{g}_0 = \mathrm{Lie}(\boldsymbol{G}_0)$ and $\mathfrak{i}_0 = \mathrm{Lie}(\boldsymbol{I}_0)$ their Lie algebras. Then* (1.8) *establishes a one-to-one correspondence between $\boldsymbol{G}_0$-homogeneous CR structures on $M = \boldsymbol{G}_0/\boldsymbol{I}_0$ and complex $\mathrm{Ad}_{\mathfrak{g}}(\boldsymbol{I}_0)$-invariant Lie subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ with $\mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{i}_0$.* □

This lead us to introduce the notion of a *CR algebra* in [28].

DEFINITION 1.4. – *A CR* algebra *is a pair $(\mathfrak{g}_0, \mathfrak{q})$, consisting of a real Lie algebra $\mathfrak{g}_0$ and of a complex Lie subalgebra $\mathfrak{q}$ of its complexification $\mathfrak{g}$, such that the quotient $\mathfrak{g}_0/(\mathfrak{q} \cap \mathfrak{g}_0)$ is finite dimensional. The real Lie subalgebra $\mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0$ is called the* isotropy subalgebra *of $(\mathfrak{g}_0, \mathfrak{q})$.*

If M is a $\boldsymbol{G}_0$-homogeneous CR manifold and $\mathfrak{q}$ is defined by (1.8), *we say that the CR algebra $(\mathfrak{g}_0, \mathfrak{q})$ is* associated *with M.*

REMARK 1.5. – The CR-dimension and CR-codimension of M can be computed in terms of its associated CR algebra $(\mathfrak{g}_0, \mathfrak{q})$. We have indeed

$$CR\text{-}\dim M = \dim_{\mathbb{C}} \mathfrak{q} - \dim_{\mathbb{C}}(\mathfrak{q} \cap \bar{\mathfrak{q}}), \tag{1.9}$$

$$CR\text{-}\mathrm{codim}\, M = \dim_{\mathbb{C}} \mathfrak{g} - \dim_{\mathbb{C}}(\mathfrak{q} + \bar{\mathfrak{q}}). \tag{1.10}$$

The CR algebra $(\mathfrak{g}_0, \mathfrak{q})$, is *totally real* when $CR\text{-}\dim M = 0$, *totally complex* when $CR\text{-}\mathrm{codim}\, M = 0$.

The scalar and vector valued Levi forms of a $\boldsymbol{G}_0$-homogeneous CR manifolds can be computed in terms of the Lie product of $\mathfrak{g}$, by using $\boldsymbol{G}_0$-left-invariant vector fields. Indeed, for $p \in M$, $\xi \in H^0_p M$, we have

$$\mathcal{L}_\xi(z_1, z_2) = -i\pi^*(\xi)([Z_1^*, \bar{Z}_2^*]) \text{ if } Z_1, Z_2 \in \mathfrak{q}, \text{ and } \pi_*(Z_i^*)_p = z_i, \tag{1.11}$$

$$\mathcal{L}_p(z) = \varpi_p(\pi_*(i[\bar{Z}^*, Z^*])) \quad \text{if } Z \in \mathfrak{q}, \text{ and } \pi_*(Z^*)_p = z, \tag{1.12}$$

$$\text{for } z, z_1, z_2 \in T^{0,1}_p M, \ \xi \in H^0_p M;$$

here Z^*, Z_1^*, Z_2^* are the left invariant vector fields of $Z, Z_1, Z_2 \in \mathfrak{q}$.

The natural isomorphism between $T_{p_0}M/H_{p_0}M$ and the quotient $\mathfrak{e} = \mathfrak{g}_0/(\{\mathfrak{q} + \bar{\mathfrak{q}}\} \cap \mathfrak{g}_0)$ makes $\mathcal{L}_{p_0}(z)$ correspond to the projection of $i[\bar{Z}, Z]$ into $\mathfrak{e}$.

DEFINITION 1.6. – *Consider a CR algebra* $(\mathfrak{g}_0, \mathfrak{q})$. *Let* $\mathfrak{Lie}_{\mathbb{C}}(\mathfrak{g})$ *be the set of complex Lie subalgebras of* $\mathfrak{g}$. *We recall that* $(\mathfrak{g}_0, \mathfrak{q})$ *is called:*

fundamental	*if*	$\mathfrak{q}' \in \mathfrak{Lie}_{\mathbb{C}}(\mathfrak{g}),\ \mathfrak{q} + \bar{\mathfrak{q}} \subset \mathfrak{q}' \Longrightarrow \mathfrak{q}' = \mathfrak{g}$,
weakly nondegenerate	*if*	$\mathfrak{q}' \in \mathfrak{Lie}_{\mathbb{C}}(\mathfrak{g}),\ \mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}} \Longrightarrow \mathfrak{q}' = \mathfrak{q}$,
Levi-nondegenerate	*if*	$\{Z \in \mathfrak{q} \mid \mathrm{ad}(Z)(\bar{\mathfrak{q}}) \subset \mathfrak{q} + \bar{\mathfrak{q}}\} = \mathfrak{q} \cap \bar{\mathfrak{q}}$,
effective	*if*	*no nontrivial ideal of* $\mathfrak{g}_0$ *is contained in* $\mathfrak{i}_0$.

If M is a $\boldsymbol{G}_0$-homogeneous *CR* manifold with associated *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$, the above properties are related to the *CR* geometry of M (see e.g. [1]) by:

(1) $(\mathfrak{g}_0, \mathfrak{q})$ is fundamental if and only if M is of finite type in the sense of Bloom and Graham (see [6]).

(2) $(\mathfrak{g}_0, \mathfrak{q})$ is Levi-nondegenerate if and only if the vector valued Levi form of M is nondegenerate. Levi-nondegeneracy implies weak nondegeneracy.

(3) $(\mathfrak{g}_0, \mathfrak{q})$ is fundamental and weakly nondegenerate if and only if the group of germs of *CR* diffeomorphisms at $p_0 \in M$ stabilizing p_0 is a finite dimensional Lie group, i.e. if and only if M is holomorphically nondegenerate (see e.g. [5], [13]).

(4) A fundamental $(\mathfrak{g}_0, \mathfrak{q})$ is weakly degenerate if and only if there exists a local $\boldsymbol{G}_0$-equivariant *CR* fibration $M \to M'$, with nontrivial complex fibers.

(5) Effectiveness means that the normal subgroups of $\boldsymbol{G}_0$ contained in the isotropy $\boldsymbol{I}_0$ are discrete.

Let $\mathfrak{g}_0, \mathfrak{g}'_0$ be real Lie algebras and $\mathfrak{q}, \mathfrak{q}'$ complex Lie subalgebras of their complexifications $\mathfrak{g}, \mathfrak{g}'$. A Lie algebra homomorphism $\phi_0 : \mathfrak{g}_0 \to \mathfrak{g}'_0$ is a *CR* algebra morphism from $(\mathfrak{g}_0, \mathfrak{q})$ to $(\mathfrak{g}'_0, \mathfrak{q}')$ if the complexification ϕ of ϕ_0 transforms $\mathfrak{q}$ into a subalgebra of $\mathfrak{q}'$. The pair (ϕ_0, ϕ) is

a CR algebra immersion if $\quad \phi^{-1}(\mathfrak{q}' \cap \bar{\mathfrak{q}}') = \mathfrak{q} \cap \bar{\mathfrak{q}},\ \ \phi^{-1}(\mathfrak{q}') = \mathfrak{q}$,

a CR algebra submersion if $\quad \phi(\mathfrak{g}) + \mathfrak{q}' \cap \bar{\mathfrak{q}}' = \mathfrak{g}',\ \ \phi(\mathfrak{q}) + \mathfrak{q}' \cap \bar{\mathfrak{q}}' = \mathfrak{q}'$,

a CR algebra local isomorphism if it is at the same time

a CR algebra immersion and submersion.

The *CR* algebra $(\mathfrak{g}_0'', \mathfrak{q}'')$ with $\mathfrak{g}_0'' = \phi_0^{-1}(\mathfrak{q}' \cap \mathfrak{g}_0')$ and $\mathfrak{q}'' = \mathfrak{q} \cap \phi^{-1}(\mathfrak{q}' \cap \bar{\mathfrak{q}}')$ is the *fiber* of $(\phi_0, \phi) : (\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0', \mathfrak{q}')$.

When $\mathfrak{g}_0 = \mathfrak{g}_0'$, $\mathfrak{q} \subset \mathfrak{q}'$, and ϕ_0 is the identity, the corresponding morphism $(\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0, \mathfrak{q}')$ is said to be $\mathfrak{g}_0$-*equivariant* (see [28]).

If M and M' are homogeneous *CR* manifolds with associated *CR* algebras $(\mathfrak{g}_0, \mathfrak{q})$, $(\mathfrak{g}_0', \mathfrak{q}')$, local *CR* maps that are local *CR* immersions, submersions or diffeomorphisms, correspond to algebraic *CR* morphisms of their *CR* algebras that are *CR* algebras immersions, submersions, or local isomorphisms, respectively, and vice versa.

For later reference, it is convenient to restate [28, Lemma 5.1] in the following form.

PROPOSITION 1.7. – *Let $\mathbf{I}_0 \subset \mathbf{I}_0'$ be closed subgroups of a Lie group $\mathbf{G}_0$. Let $\mathfrak{i}_0, \mathfrak{i}_0', \mathfrak{g}_0$ be the Lie algebras of $\mathbf{I}_0, \mathbf{I}_0', \mathbf{G}_0$, and $\mathfrak{i}, \mathfrak{i}', \mathfrak{g}$ their complexifications, respectively. Let $(\mathfrak{g}_0, \mathfrak{q})$ be a CR algebra, defining a $\mathbf{G}_0$-invariant CR structure on $M = \mathbf{G}_0/\mathbf{I}_0$. Then a necessary and sufficient condition for the existence of a $\mathbf{G}_0$-invariant CR structure on $M' = \mathbf{G}_0/\mathbf{I}_0'$ making the $\mathbf{G}_0$-equivariant map $\pi : M \to M'$ a CR submersion is that:*

$$\mathfrak{q}' = \mathfrak{q} + \mathfrak{i}' \text{ is a Lie algebra}, \quad \text{and} \quad \mathfrak{q}' \cap \mathfrak{g}_0 = \mathfrak{i}_0'. \tag{1.13}$$

When (1.13) *holds, it defines the CR algebra $(\mathfrak{g}_0, \mathfrak{q}')$ at $p_0' = [\mathbf{I}_0']$ which defines the unique $\mathbf{G}_0$-homogeneous CR structure on M' for which $M \xrightarrow{\pi} M'$ is a CR submersion.*

1.3 – *The J-property.*

Let M be a *CR* manifold. Its *partial complex structure* is the vector bundle isomorphism $J : HM \to HM$ that associates to $X_p \in H_pM$ the vector $JX_p \in H_pM$ for which $X_p + iJX_p \in T_p^{0,1}M$.

Let M be a $\mathbf{G}_0$-homogeneous *CR* manifold, with *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$ at $p_0 \in M$, and set $\mathfrak{V}_0 = \{\operatorname{Re} Z \mid Z \in \mathfrak{q}\}$. The partial complex structure of M yields a complex structure on $\mathfrak{V}_0/\mathfrak{i}_0$, via its canonical identification with $H_{p_0}M$. This is the *partial complex structure of* $(\mathfrak{g}_0, \mathfrak{q})$.

DEFINITION 1.8. – *We say that $(\mathfrak{g}_0, \mathfrak{q})$ has the J-property if $J \in \operatorname{Der}(\mathfrak{g}_0)$ can be chosen in such a way that*

$$J(\mathfrak{i}_0) \subset \mathfrak{i}_0, \quad X + iJ(X) \in \mathfrak{q}, \quad \forall X \in \mathfrak{V}_0. \tag{1.14}$$

We say that a CR algebra $(\mathfrak{g}_0, \mathfrak{q})$ has the weak-J-property if there is $J \in \operatorname{Der}(\mathfrak{g}_0)$ such that, for $\Upsilon = \operatorname{Ad}(\exp(\pi J/2))$,

$$\Upsilon(\mathfrak{i}_0) = \mathfrak{i}_0, \quad X + i\,\Upsilon(X) \in \mathfrak{q}, \quad \forall X \in \mathfrak{V}_0. \tag{1.15}$$

If $J \in \mathrm{Der}(\mathfrak{g}_0)$ *satisfies* (1.14), *then* $\Upsilon = \mathrm{Ad}(\exp(\pi J/2))$ *satisfies* (1.15). *Hence the first condition is stronger than the second.*

REMARK 1.9. – Conditions (1.14) and (1.15) can also be expressed in terms of the complexifications of J, Υ. Namely, denoting by the same letter also their complexifications, they are equivalent to

$$(1.14)' \qquad J(\mathfrak{q}) \subset \mathfrak{q}, \quad Z - iJ(Z) \in \mathfrak{q} \cap \bar{\mathfrak{q}}, \ \forall Z \in \mathfrak{q},$$

$$(1.15)' \qquad \Upsilon(\mathfrak{q}) = \mathfrak{q}, \quad Z - i\Upsilon(Z) \in \mathfrak{q} \cap \bar{\mathfrak{q}}, \ \forall Z \in \mathfrak{q}.$$

For a map $A \in \mathfrak{gl}(\mathfrak{g}_0)$, we denote by A_s and A_n its semisimple and nilpotent parts, respectively. If $A \in \mathrm{Der}(\mathfrak{g}_0)$, then also A_s and A_n are derivations of $\mathfrak{g}_0$ (see e.g [19, § 4.2, Lemma b]).

PROPOSITION 1.10. – *Let* $(\mathfrak{g}_0, \mathfrak{q})$ *be a CR algebra, and* $J \in \mathrm{Der}(\mathfrak{g}_0)$*. If* J *satisfies* (1.14), *then also* J_s *satisfies* (1.14). *If* $\Upsilon = \mathrm{Ad}(\exp(\pi J/2))$ *satisfies* (1.15), *then also* $\Upsilon_s = \mathrm{Ad}(\exp(\pi J_s/2))$ *satisfies* (1.15).

PROOF. – Indeed, $\mathrm{Ad}(\pi J_s/2)$ is the semisimple part of $\mathrm{Ad}(\pi J/2)$. Since J_s and Υ_s are polynomials of J, Υ, respectively, $(1.14)'$ for J implies $(1.14)'$ for J_s, and likewise $(1.15)'$ for Υ implies $(1.15)'$ for Υ_s. □

As a consequence, we can always assume in Definition 1.8 that J be a semisimple derivation of $\mathfrak{g}_0$.

1.4 – *Symmetric CR manifolds.*

Let M be a *CR* manifold, with partial complex structure J. A Riemannian metric g on M is *CR-compatible* if $g(JX_p, JX_p) = g(X_p, X_p)$ for all $p \in M$ and $X_p \in H_pM$. Let $\Theta(M)$ be the Lie algebra of real vector fields generated by $\Gamma(M, HM)$ and $\Theta_pM = \{X_p \mid X \in \Theta(M)\}$. Note that $\Theta_pM = T_pM$ when M is of finite type in the sense of Bloom and Graham. Denote by $\Theta_p^{\perp}M$ the orthogonal of Θ_pM in T_pM for the Riemannian metric g.

DEFINITION 1.11 (see [20]). – *Let* M *be a CR manifold, with a CR-compatible Riemannian structure. We say that* M *is CR-symmetric if, for each* $p \in M$*, there is an isometry* $\sigma_p : M \to M$ *that fixes* p*, is a CR map, and whose differential restricts to* $-\mathrm{Id}$ *on* $H_pM \oplus \Theta_p^{\perp}M$.

In [20, Proposition 3.6] the *CR*-isometries of a symmetric *CR*-manifold M are proved to form a transitive group $\boldsymbol{G}_0$ of transformations of M.

Given a *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$, let $\mathfrak{q}^\natural$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{q}+\bar{\mathfrak{q}}$.

We recall that a subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}_0$ is compact if the Killing form of $\mathfrak{g}_0$ is negative definite on $\mathfrak{k}_0$. We say that a subalgebra $\mathfrak{i}_0$ of $\mathfrak{g}_0$ is almost compact if there exists a decomposition $\mathfrak{i}_0 = \mathfrak{k}_0 \oplus \mathfrak{t}_0$ with $\mathfrak{k}_0$ compact in $\mathfrak{g}_0$ and $\mathfrak{t}_0$ contained in the kernel of the Killing form of $\mathfrak{g}_0$.

DEFINITION 1.12. – *We say that* $(\mathfrak{g}_0, \mathfrak{q})$ *is CR-symmetric if* $\mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0$ *is almost compact in* $\mathfrak{g}_0$*, and there exists an involution* λ *of* $\mathfrak{g}$ *with*

$$\begin{gathered}\lambda(\mathfrak{g}_0) = \mathfrak{g}_0, \quad \ker(\mathrm{Id}-\lambda) \subset \mathfrak{q}^\natural, \quad \lambda(\mathfrak{q}) = \mathfrak{q}, \\ Z + \lambda(Z) \in \mathfrak{q} \cap \bar{\mathfrak{q}}, \ \forall Z \in \mathfrak{q}.\end{gathered} \tag{1.16}$$

Conditions (1.16) imply that

$$[Z_1, Z_2] \in \mathfrak{q} \cap \bar{\mathfrak{q}}, \quad \forall Z_1, Z_2 \in \mathfrak{q}. \tag{1.17}$$

The involution λ is equivalent to the datum of a $\mathbb{Z}_2$-gradation

$$\mathfrak{g} = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}, \quad [\mathfrak{g}_{(i)}, \mathfrak{g}_{(j)}] \subset \mathfrak{g}_{(i+j)}, \tag{1.18}$$

where (i) denotes the congruence class of $i \in \mathbb{Z}$ in $\mathbb{Z}_2$, *compatible* with $(\mathfrak{g}_0, \mathfrak{q})$. Compatibility means that:

$$\begin{cases} \mathfrak{g}_{(0)} \subset \mathfrak{q}^\natural, \quad \mathfrak{q} \cap \mathfrak{g}_{(0)} \subset \mathfrak{q} \cap \bar{\mathfrak{q}}, \\ \mathfrak{q} = (\mathfrak{q} \cap \mathfrak{g}_{(0)}) \oplus (\mathfrak{q} \cap \mathfrak{g}_{(1)}), \\ \mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{g}_{(0)}) \oplus (\mathfrak{g}_0 \cap \mathfrak{g}_{(1)}). \end{cases} \tag{1.19}$$

The involution λ and the $\mathbb{Z}_2$-gradation (1.18) are related by

$$\mathfrak{g}_{(0)} = \{Z \in \mathfrak{g} \mid \lambda(Z) = Z\}, \quad \mathfrak{g}_{(1)} = \{Z \in \mathfrak{g} \mid \lambda(Z) = -Z\}, \tag{1.20}$$

and (1.16), (1.19) are equivalent to define the *CR*-symmetry of $(\mathfrak{g}_0, \mathfrak{q})$.

PROPOSITION 1.13. – *Let* $(\mathfrak{g}_0, \mathfrak{q})$ *be a fundamental CR algebra with* $\mathfrak{i}_0$ *almost compact, and having the weak-J-property. If* $J(\mathfrak{i}_0) = 0$*, then* $(\mathfrak{g}_0, \mathfrak{q})$ *is CR-symmetric.*

PROOF. – Indeed, by the assumptions, the automorphism $\lambda = \mathrm{Ad}(\exp(\pi J))$ is an involution of $\mathfrak{g}$ that satisfies (1.16). □

PROPOSITION 1.14. – *Let* M *be a CR-manifold. Assume that* M *is CR-symmetric for a CR-compatible Riemannian structure. Let* $\boldsymbol{G}_0$ *be the transitive group of CR-isometries of* M*, and* $(\mathfrak{g}_0, \mathfrak{q})$ *the corresponding CR algebra of* M *at* $p_0 \in M$*. Then* $(\mathfrak{g}_0, \mathfrak{q})$ *is CR-symmetric.*

Vice versa, if M is a $\boldsymbol{G}_0$-homogeneous CR manifold, having at a point $p_0 \in M$ a CR algebra $(\mathfrak{g}_0, \mathfrak{q})$ which is CR-symmetric, and the analytic subgroup tangent to $\mathfrak{i}_0$ is compact, then there is a compatible Riemannian metric on M for which M is CR-symmetric. □

2. – Levi-Malcev and Jordan-Chevalley fibrations.

2.1 – $\boldsymbol{A}_0$*-fibrations.*

Let $\boldsymbol{G}_0$ be a Lie group, $\mathfrak{g}_0$ its Lie algebra, $\mathfrak{a}_0$ an ideal of $\mathfrak{g}_0$, and $\boldsymbol{A}_0$ the corresponding analytic normal subgroup of $\boldsymbol{G}_0$.

DEFINITION 2.1. – *Let $M = \boldsymbol{G}_0/\boldsymbol{I}_0$ be a homogeneous space of $\boldsymbol{G}_0$. If the subgroup $\boldsymbol{A}_0\boldsymbol{I}_0$ is closed in $\boldsymbol{G}_0$, we call the $\boldsymbol{G}_0$-equivariant fibration*

$$M = \boldsymbol{G}_0/\boldsymbol{I}_0 \xrightarrow{\pi} M' = \boldsymbol{G}_0/(\boldsymbol{A}_0\boldsymbol{I}_0) \tag{2.1}$$

the $\boldsymbol{A}_0$-fibration of M.

Assuming that M admits an $\boldsymbol{A}_0$-fibration (2.1), we will discuss the existence of compatible $\boldsymbol{G}_0$-homogeneous *CR* structures on $M = \boldsymbol{G}_0/\boldsymbol{I}_0$ and $M' = \boldsymbol{G}_0/(\boldsymbol{A}_0\boldsymbol{I}_0)$. Denote by $\mathfrak{a}$ the complexification of $\mathfrak{a}_0$.

PROPOSITION 2.2. – *Let $(\mathfrak{g}_0, \mathfrak{q})$, with $\mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{i}_0$, be a CR algebra defining a $\boldsymbol{G}_0$-homogeneous CR structure on M, and assume that the subgroup $\boldsymbol{A}_0\boldsymbol{I}_0$ is closed.*

A necessary and sufficient condition for the existence of a $\boldsymbol{G}_0$-homogeneous CR structure on $M' = \boldsymbol{G}/(\boldsymbol{A}_0\boldsymbol{I}_0)$, making the $\boldsymbol{A}_0$-fibration (2.1) *a CR map is that:*

$$\mathfrak{q} \cap \bar{\mathfrak{q}} + \mathfrak{a} = (\mathfrak{q} + \mathfrak{a}) \cap (\bar{\mathfrak{q}} + \mathfrak{a}). \tag{2.2}$$

Assume that (2.2) *is satisfied and define the CR structure on M' by $(\mathfrak{g}_0, \mathfrak{q}')$, with $\mathfrak{q}' = \mathfrak{q} + \mathfrak{a}$. Then:*

(1) (2.1) *is a $\boldsymbol{G}_0$-equivariant CR fibration.*

(2) *Its typical fiber F is the $\boldsymbol{A}_0$-homogeneous manifold $\boldsymbol{A}_0/(\boldsymbol{A}_0 \cap \boldsymbol{I}_0)$, having an $\boldsymbol{A}_0$-homogeneous CR structure defined by the CR algebra $(\mathfrak{a}_0, \mathfrak{q} \cap \mathfrak{a})$.*

PROOF. – By Proposition 1.3, the $\boldsymbol{G}_0$-homogeneous *CR* structures on M' are in one-to-one correspondence with the *CR* algebras $(\mathfrak{g}_0, \mathfrak{q}')$, with isotropy $\mathfrak{i}'_0 = \mathfrak{q}' \cap \mathfrak{g}_0$ equal to $(\mathfrak{i}_0 + \mathfrak{a}_0)$. The map $\pi : M \to M'$ is *CR* if $\mathfrak{q} \subset \mathfrak{q}'$. Thus $(\mathfrak{i}_0 + \mathfrak{a}_0) \subset (\mathfrak{q} \cap \bar{\mathfrak{q}} + \mathfrak{a}) \subset \mathfrak{q}'$, and $\mathfrak{q} + \mathfrak{a} \subset \mathfrak{q}'$. By Proposition 1.7, the map $\pi : M \to M'$ is a *CR* submersion if and only if the last inclusion is an equality. Finally (1) and (2) follow by [28, § 5]. □

PROPOSITION 2.3. – *We keep the notation above. Assume that* $(\mathfrak{g}_0, \mathfrak{q})$ *has the weak-J-property and that* $\mathfrak{a}_0$ *is* Υ*-invariant. Then*:

(1) *Condition* (2.2) *is satisfied.*

(2) *The basis* $(\mathfrak{g}_0, \mathfrak{q}+\mathfrak{a})$ *and the fiber* $(\mathfrak{a}_0, \mathfrak{a}\cap\mathfrak{q})$ *of the* $\boldsymbol{A}_0$*-fibration enjoy the weak-J-property.*

If we assume that $(\mathfrak{g}_0, \mathfrak{q})$ *has the J-property, then both the basis and the fiber of the* $\boldsymbol{A}_0$*-fibration enjoy the J-property.*

PROOF. – Let $J \in \mathrm{Der}(\mathfrak{g}_0)$ be such that $\Upsilon = \mathrm{Ad}(\exp(\pi J/2))$ satisfies (1.15).

(1) We only need to prove the inclusion $(\mathfrak{q}+\mathfrak{a})\cap\mathfrak{g}_0 \subset \mathfrak{i}_0 + \mathfrak{a}_0$. An element A of $(\mathfrak{q}+\mathfrak{a})\cap\mathfrak{g}_0$ is a sum $A = (X+iY)+(U-iY)$, with $X, Y, U \in \mathfrak{g}_0$, $X+iY \in \mathfrak{q}$ and $U, Y \in \mathfrak{a}_0$. Since both $Y - iX$ and $Y + i\Upsilon(Y)$ belong to $\mathfrak{q}$, we obtain that $X + \Upsilon(Y) \in \mathfrak{q}\cap\mathfrak{g}_0 = \mathfrak{i}_0$. Moreover, $\Upsilon(Y) \in \mathfrak{a}_0$, because $\mathfrak{a}_0$ is Υ-invariant. Hence $A = (X+\Upsilon(Y)) + (U - \Upsilon(Y)) \in \mathfrak{i}_0 + \mathfrak{a}_0$.

(2) The subalgebras $\mathfrak{q}+\mathfrak{a}$ and $(\mathfrak{q}\cap\bar{\mathfrak{q}})+\mathfrak{a}$ are Υ-invariant. Thus Υ yields multiplication by i on the quotient $(\mathfrak{q}+\mathfrak{a})/((\mathfrak{q}\cap\bar{\mathfrak{q}})+\mathfrak{a})$. This proves the statement for the base. The statement for the fiber is trivial.

The last statement can be obtained by repeating with minor changes the arguments used above for the proof of (2). □

We will apply the results above to the cases where $\mathfrak{a}_0$ is either the radical or the nilpotent radical of $\mathfrak{g}_0$.

2.2 – *The Levi-Malcev fibration.*

Let $\mathfrak{g}_0$ be a real Lie algebra and $\mathfrak{r}_0$ its radical. The Levi-Malcev decomposition of $\mathfrak{g}_0$ has the form

$$\mathfrak{g}_0 = \mathfrak{r}_0 \oplus \mathfrak{s}_0, \tag{2.3}$$

where $\mathfrak{s}_0$ is a semisimple Levi factor of $\mathfrak{g}_0$, i.e. a Lie subalgebra of $\mathfrak{g}_0$ isomorphic to the quotient $\mathfrak{g}_0/\mathfrak{r}_0$.

Let $\boldsymbol{G}_0$ be a Lie group with Lie algebra $\mathfrak{g}_0$. Its radical $\boldsymbol{R}_0$ is its maximal connected solvable subgroup, and equals its analytic Lie subgroup with Lie algebra $\mathfrak{r}_0$.

DEFINITION 2.4. – *Let* $M = \boldsymbol{G}_0/\boldsymbol{I}_0$ *be a homogeneous space of* $\boldsymbol{G}_0$. *If* $\boldsymbol{R}_0\boldsymbol{I}_0$ *is a closed subgroup of* $\boldsymbol{G}_0$, *we call the* $\boldsymbol{G}_0$*-equivariant fibration*

$$M = \boldsymbol{G}_0/\boldsymbol{I}_0 \xrightarrow{\pi} M' = \boldsymbol{G}_0/\boldsymbol{R}_0\boldsymbol{I}_0 \tag{2.4}$$

the Levi-Malcev fibration of M.

EXAMPLE 2.5. – Not all homogeneous spaces admit a Levi-Malcev fibration. Take, for instance, $\boldsymbol{G}_0 = \boldsymbol{SU}(3) \times \mathbb{R}^+$ and $\boldsymbol{I}_0 = \{(\exp(tX), e^t) \mid t \in \mathbb{R}\}$ for $X = i\,\mathrm{diag}(\alpha, \beta, \gamma)$, with $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha + \beta + \gamma = 0$, and α, β linearly independent over $\mathbb{Q}$. Let $\boldsymbol{R}_0$ be the radical of $\boldsymbol{G}_0$. Then $\boldsymbol{I}_0$ is closed, but $\boldsymbol{R}_0\boldsymbol{I}_0 = \{(\exp(tX), e^s) \mid t, s \in \mathbb{R}\}$ is not closed in $\boldsymbol{G}_0$.

EXAMPLE 2.6. – Let $\mathfrak{s}_0$ be a semisimple real Lie algebra and V_0 a nontrivial real irreducible $\mathfrak{s}_0$-module. Let $\mathfrak{g}_0 = \mathfrak{s}_0 \oplus V_0$ be the Abelian extension of $\mathfrak{s}_0$ by V_0. Its Lie algebra structure is defined by

$$\begin{cases} [X_1 + v_1, X_2 + v_2] = [X_1, X_2] + X_1 \cdot v_2 - X_2 \cdot v_1 \\ \qquad \text{for } X_1, X_2 \in \mathfrak{s}_0,\ v_1, v_2 \in V_0. \end{cases}$$

Radical and nilradical of $\mathfrak{g}_0$ are both equal to $V_0 \simeq 0 \oplus V_0$, and $\mathfrak{s}_0 \simeq \mathfrak{s}_0 \oplus 0 \subset \mathfrak{g}_0$ is a Levi subalgebra and a reductive component of $\mathfrak{g}_0$. Then $\mathfrak{g}_0 = \mathfrak{s}_0 \oplus V_0$ is a Jordan-Chevalley and a Levi decomposition of $\mathfrak{g}_0$, at the same time.

Fix a connected semisimple Lie group $\boldsymbol{S}_0$ with Lie algebra $\mathfrak{s}_0$, to which the representation of $\mathfrak{s}_0$ on V_0 lifts. The product

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, v_1 + g_1(v_2)), \text{ for } g_1, g_2 \in \boldsymbol{S}_0,\ v_1, v_2 \in V_0,$$

defines on $\boldsymbol{G}_0 = \boldsymbol{S}_0 \ltimes V_0$ the structure of a Lie group with Lie algebra $\mathfrak{g}_0$ and radical $\boldsymbol{R}_0 = \{e_{\boldsymbol{S}_0}\} \times V_0$.

Let $\mathfrak{s}$ and V be the complexifications of $\mathfrak{s}_0$ and V_0, respectively, so that $\mathfrak{g} = \mathfrak{s} \oplus V$ is the complexification of $\mathfrak{g}_0$.

Fix a closed subgroup $\boldsymbol{A}_0$ of $\boldsymbol{S}_0$, with Lie algebra $\mathfrak{a}_0$. The stabilizer $\boldsymbol{I}_0$ of any vector $v_0 \in V_0$ in $\boldsymbol{A}_0$ is a closed subgroup of $\boldsymbol{S}_0$ and hence of $\boldsymbol{G}_0$. Let $M = \boldsymbol{G}_0/\boldsymbol{I}_0 \simeq (\boldsymbol{S}_0/\boldsymbol{I}_0) \times V_0$ be endowed with the $\boldsymbol{G}_0$-homogeneous *CR* structure defined by $(\mathfrak{g}_0, \mathfrak{q})$, for

$$\mathfrak{q} = \mathbb{C} \cdot \{X + iX \cdot v_0 \,|\, X \in \mathfrak{a}_0\} \subset \mathfrak{g}.$$

With $\mathfrak{a}$ equal to the complexification of $\mathfrak{a}_0$, we have:

$$\begin{cases} \mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0 = \mathfrak{q} \cap \mathfrak{s}_0 = \{X \in \mathfrak{a}_0 \,|\, X \cdot v_0 = 0\}, \\ \mathfrak{q} + V = \mathfrak{a} \oplus V, \\ \mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{a} \oplus (\mathfrak{a} \cdot v_0). \end{cases}$$

Hence the *CR* algebra $(\mathfrak{g}_0, \mathfrak{q} + V)$, corresponding to the basis of the $\boldsymbol{G}_0$-equivariant map $M \to N = \boldsymbol{G}_0/(\boldsymbol{A}_0 \times V_0)$, is totally real and locally *CR* isomorphic to $(\mathfrak{s}_0, \mathfrak{a})$. The fiber F is an $(\boldsymbol{A}_0 \ltimes V_0)$-homogeneous *CR* manifold, with *CR* algebra $(\mathfrak{a}_0 \oplus V_0, \mathfrak{q})$. Thus, if $\mathfrak{a}_0 \neq \mathfrak{i}_0$, there is no $\boldsymbol{G}_0$-homogeneous *CR* structure on $M' = \boldsymbol{G}_0/\boldsymbol{R}_0\boldsymbol{I}_0$ such that the fibration $M \to M'$ is an equivariant *CR* submersion.

The Levi subalgebras of $\mathfrak{g}_0$ are parametrized by the elements of V_0:

$$\mathfrak{s}_0^{(v)} = \{X + X \cdot v \mid X \in \mathfrak{s}_0\}, \quad \text{for} \quad v \in V_0.$$

We have

$$\mathfrak{s}_0^{(v)} \cap \mathfrak{q} = \{X \in \mathfrak{s}_0 \mid X(v_0) = 0,\ X(v) = 0\} \subset \mathfrak{i}_0,$$

$$\mathfrak{s}^{(v)} \cap \mathfrak{q} = \{\ Z \in \mathfrak{s}\ \mid Z(v_0) = 0,\ Z(v) =\ 0\} = \left(\mathfrak{s}_0^{(v)} \cap \mathfrak{q}\right)^{\mathbb{C}},$$

so that, for every choice of $v \in V_0$, the *CR* algebra $(\mathfrak{s}_0^{(v)}, \mathfrak{q} \cap \mathfrak{s}^{(v)})$ is totally real. Thus, if $\mathfrak{a}_0 \neq \mathfrak{i}_0$, there is no Levi factor $\mathfrak{s}_0^{(v)}$ in $\mathfrak{g}_0$ such that $(\mathfrak{s}_0^{(v)}, (\mathfrak{q} \cap \mathfrak{s}^{(v)})) \simeq (\mathfrak{g}_0, \mathfrak{q} + V)$.

In [27] the homogeneous *CR* manifolds M with $(\mathfrak{g}_0, \mathfrak{q})$ of Levi-Tanaka type were shown to admit a Levi-Malcev fibration; (2.4) is in this case a *CR* submersion with basis and fiber having both *CR* structures of Levi-Tanaka type. Example 2.6 shows that, even when it does exist, we cannot expect to find, on the basis M' of the Levi-Malcev fibration (2.4) of a $\boldsymbol{G}_0$-homogeneous *CR* manifold M, a $\boldsymbol{G}_0$-homogeneous *CR* structure that makes (2.4) a *CR* map. We have, by Proposition 2.2:

COROLLARY 2.7. – *Assume that the $\boldsymbol{G}_0$-homogeneous CR structure of $M = \boldsymbol{G}_0/\boldsymbol{I}_0$ is described by the CR algebra* $(\mathfrak{g}_0, \mathfrak{q})$ *and that M admits the Levi-Malcev fibration* (2.4). *Then a necessary and sufficient condition for the existence of a $\boldsymbol{G}_0$-homogeneous CR structure on M', making* (2.4) *a CR map, is that*

$$\mathfrak{q} \cap \bar{\mathfrak{q}} + \mathfrak{r} = (\mathfrak{q} + \mathfrak{r}) \cap (\bar{\mathfrak{q}} + \mathfrak{r}). \tag{2.5}$$

Moreover we obtain:

THEOREM 2.8. – *Suppose that* (2.5) *is valid, and consider on the basis M' of the Levi-Malcev fibration* (2.4) *the $\boldsymbol{G}_0$-homogeneous CR structure defined by* $(\mathfrak{g}_0, \mathfrak{q}')$, *with* $\mathfrak{q}' = \mathfrak{q} + \mathfrak{r}$. *Then:*

(1) *M' is an $\boldsymbol{S}_0$-homogeneous CR manifold $M' \simeq \boldsymbol{S}_0/(\boldsymbol{S}_0 \cap \boldsymbol{R}_0\boldsymbol{I}_0)$, with CR algebra* $(\mathfrak{s}_0, \mathfrak{s} \cap \mathfrak{q}')$.

(2) *The fiber of* (2.4) *is the solvmanifold $F \simeq \boldsymbol{R}_0/(\boldsymbol{R}_0 \cap \boldsymbol{I}_0)$, with $\boldsymbol{R}_0$-homogeneous CR structure defined by* $(\mathfrak{r}_0, \mathfrak{r} \cap \mathfrak{q})$.

If $(\mathfrak{g}_0, \mathfrak{q})$ *has the weak-J-property (resp. the J-property) then:*

(3) *Condition* (2.5) *is satisfied.*

(4) *Both the basis M' and the fiber F of the Levi-Malcev fibration enjoy the weak-J-property (resp. the J-property).*

PROOF. – The result follows from Propositions 2.2 and 2.3, after noticing that $\mathfrak{r}_0$ is a characteristic ideal, hence J-invariant. □

2.3 – *The Jordan-Chevalley fibration.*

An algebraic group $\boldsymbol{G}$ over a field $\Bbbk$ always contains a maximal normal solvable subgroup $\boldsymbol{R}^*$. The connected component $\boldsymbol{R}$ of the identity in $\boldsymbol{R}^*$ is the *radical* of $\boldsymbol{G}$. The set $\boldsymbol{N}$ of unipotent elements of $\boldsymbol{R}$ is a connected normal subgroup of $\boldsymbol{G}$, called the *unipotent radical of* $\boldsymbol{G}$. The algebraic group $\boldsymbol{G}$ is *reductive* when its unipotent radical is trivial.

If the field $\Bbbk$ is *perfect* ([1]), any algebraic group $\boldsymbol{G}$ over $\Bbbk$ admits a *Jordan-Chevalley decomposition* (see e.g. [7, 12, 29]), i.e. there is a maximal reductive subgroup $\boldsymbol{L}$ of $\boldsymbol{G}$ such that

$$\boldsymbol{G} = \boldsymbol{N} \rtimes \boldsymbol{L}. \tag{2.6}$$

For the proof of the following Lemma, see e.g. [18, Ch. VII, Lemma 1.4].

Lemma 2.9. – *Let* $\mathfrak{g} \subset \mathfrak{gl}(n, \Bbbk)$ *be a linear Lie algebra,* $\mathfrak{n}$ *an ideal and* $\mathfrak{a}$ *a subalgebra of* $\mathfrak{g}$*. If all the elements of* $\mathfrak{n} \cup \mathfrak{a}$ *are nilpotent, then all the elements of* $\mathfrak{a} + \mathfrak{n}$ *are nilpotent.* □

Proposition 2.10. – *Let* $\boldsymbol{G}$ *be an algebraic group over a perfect field* $\Bbbk$ *and* $\boldsymbol{N}$ *its unipotent radical. If* $\boldsymbol{I}$ *is an algebraic subgroup of* $\boldsymbol{G}$*, then also* $\boldsymbol{N}\boldsymbol{I}$ *is an algebraic subgroup of* $\boldsymbol{G}$*.*

Proof. – Let $\mathfrak{n}$ and $\mathfrak{i}$ be the Lie algebras of $\boldsymbol{N}$ and $\boldsymbol{I}$, respectively. Let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{I}$ and $\mathfrak{u}$ its Lie algebra. By Lemma 2.9, the sum $\mathfrak{n}' = \mathfrak{u} + \mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$. The set

$$\boldsymbol{G}'' = \{g \in \boldsymbol{G} \mid \mathrm{Ad}(g)(\mathfrak{n}') = \mathfrak{n}'\} \tag{2.7}$$

is an algebraic subgroup of $\boldsymbol{G}$ containing $\boldsymbol{I}$. Let $\mathfrak{g}''$ be its Lie algebra and $\mathfrak{n}''$, which is contained in $\mathfrak{g}''$, that of the unipotent radical $\boldsymbol{N}''$ of $\boldsymbol{G}''$. The analytic subgroup $\boldsymbol{N}'$ corresponding to $\mathfrak{n}'$ is a normal subgroup of $\boldsymbol{G}''$ consisting of unipotent elements. Hence $\boldsymbol{N}' \subset \boldsymbol{N}''$, and therefore $\boldsymbol{N}'$ is Zariski-closed and algebraic in both $\boldsymbol{G}''$ and $\boldsymbol{G}$.

If $\boldsymbol{I} = \boldsymbol{L}' \ltimes \boldsymbol{U}$ is a Jordan-Chevalley decomposition of $\boldsymbol{I}$, then we obtain a decomposition $\boldsymbol{N}\boldsymbol{I} = \boldsymbol{L}' \ltimes \boldsymbol{N}'$. Hence $\boldsymbol{N}\boldsymbol{I}$ is algebraic, being a semidirect product of algebraic subgroups of $\boldsymbol{G}$. □

Definition 2.11. – *Let* $\boldsymbol{G}_0$ *be a real algebraic group, with unipotent radical* $\boldsymbol{N}_0$*. If* $\boldsymbol{I}_0$ *is an algebraic subgroup of* $\boldsymbol{G}_0$*, then by Proposition* 2.10 *also* $\boldsymbol{N}_0\boldsymbol{I}_0$ *is*

([1]) This means that all algebraic extensions of $\Bbbk$ are separable. This is equivalent to the fact that either $\Bbbk$ has characteristic 0, or, having positive characteristic p, every element of $\Bbbk$ admits a p-th root in $\Bbbk$.

algebraic. The $\boldsymbol{G}_0$*-equivariant fibration*

$$M = \boldsymbol{G}_0/\boldsymbol{I}_0 \longrightarrow M' = \boldsymbol{G}_0/\boldsymbol{N}_0\boldsymbol{I}_0 \tag{2.8}$$

is called the Jordan-Chevalley *fibration of M.*

In this setting, Propositions 2.2 and 2.3 yield the following result.

THEOREM 2.12. – *Let* $\boldsymbol{G}_0$ *be a real linear algebraic group and* $\boldsymbol{N}_0$ *its unipotent radical. We denote by* $\mathfrak{g}_0$, $\mathfrak{n}_0$ *the Lie algebras of* $\boldsymbol{G}_0$, $\boldsymbol{N}_0$, *and by* $\mathfrak{g}$, $\mathfrak{n}$ *their complexifications, respectively.*

Let $M = \boldsymbol{G}_0/\boldsymbol{I}_0$, *for an algebraic subgroup* $\boldsymbol{I}_0$, *have a* $\boldsymbol{G}_0$*-invariant CR structure defined by the CR algebra* $(\mathfrak{g}_0, \mathfrak{q})$.

A necessary and sufficient condition in order that there exists a $\boldsymbol{G}_0$*-homogeneous CR structure on the basis* $M' = \boldsymbol{G}/\boldsymbol{N}_0\boldsymbol{I}_0$ *of the Jordan-Chevalley fibration, making* (2.8) *a CR map is that:*

$$\mathfrak{q} \cap \bar{\mathfrak{q}} + \mathfrak{n} = (\mathfrak{q} + \mathfrak{n}) \cap (\bar{\mathfrak{q}} + \mathfrak{n}). \tag{2.9}$$

Assume that (2.9) *is satisfied and consider on* M' *the CR structure defined by* $(\mathfrak{g}_0, \mathfrak{q}')$ *with* $\mathfrak{q}' = \mathfrak{q} + \mathfrak{n}$. *Then:*

(1) (2.8) *is a CR fibration.*

(2) *Its typical fiber* F *is the nilmanifold* $\boldsymbol{N}_0/(\boldsymbol{N}_0 \cap \boldsymbol{I}_0)$, *with an* $\boldsymbol{N}_0$*-homogeneous CR structure defined by the CR algebra* $(\mathfrak{n}_0, \mathfrak{q} \cap \mathfrak{n})$.

(3) *The basis* $M' = \boldsymbol{G}_0/\boldsymbol{N}_0\boldsymbol{I}_0$ *is an* $\boldsymbol{L}_0$*-homogeneous CR manifold, associated with the CR algebra* $(\mathfrak{l}_0, \mathfrak{l} \cap (\mathfrak{q} + \mathfrak{n}))$, *where* $\boldsymbol{L}_0$ *is a maximal reductive subgroup of* $\boldsymbol{G}_0$, $\mathfrak{l}_0$ *its Lie algebra, and* $\mathfrak{l}$ *its complexification.*

If $(\mathfrak{g}_0, \mathfrak{q})$ *has the weak-J-property (resp. the J-property), then:*

(4) *Condition* (2.9) *is satisfied.*

(5) *Both the basis* M' *and the fiber* F *of* (2.8) *enjoy the weak-J-property (resp. the J-property).*

PROOF. – The first part of the statement is a consequence of Proposition 2.2. Finally, (1), (2), (3) follow by [28, § 5], and (4), (5) because $\mathfrak{n}_0$ is a characteristic ideal. □

3. – Attaching homogeneous *CR* manifolds to *CR* algebras.

3.1 – *CR manifolds associated to a CR algebra.*

Let us consider the question of the existence of homogeneous *CR* manifolds associated with a given *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$.

DEFINITION 3.1. – *A CR algebra* $(\mathfrak{g}_0, \mathfrak{q})$ *is* factual *if there exists a real Lie group* $\boldsymbol{G}_0$ *with Lie algebra* $\mathfrak{g}_0$*, and a closed subgroup* $\boldsymbol{I}_0$ *of* $\boldsymbol{G}_0$ *with Lie algebra* $\mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0$.

We have:

THEOREM 3.2. – *Let* $(\mathfrak{g}_0, \mathfrak{q})$ *be a CR algebra,* $\tilde{\boldsymbol{G}}_0$ *a connected and simply connected real Lie group with Lie algebra* $\mathfrak{g}_0$*, and* $\tilde{\boldsymbol{I}}_0$ *its analytic subgroup with Lie algebra* $\mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0$. *Then* $(\mathfrak{g}_0, \mathfrak{q})$ *is factual if, and only if,* $\tilde{\boldsymbol{I}}_0$ *is closed in* $\tilde{\boldsymbol{G}}_0$.

PROOF. – The statement is a special case of a general fact, only involving homogeneous spaces. If $\boldsymbol{G}_0$ is any connected Lie group with Lie algebra $\mathfrak{g}_0$, containing a closed subgroup $\boldsymbol{I}_0$ with Lie algebra $\mathfrak{i}_0$, then the universal covering $\tilde{M}$ of the homogeneous manifold $M = \boldsymbol{G}_0/\boldsymbol{I}_0$ is $\tilde{\boldsymbol{G}}_0$-homogeneous, and is of the form $\tilde{\boldsymbol{G}}_0/\tilde{\boldsymbol{I}}_0$, for the analytic Lie subgroup $\tilde{\boldsymbol{I}}_0$ of $\tilde{\boldsymbol{G}}_0$ corresponding to the Lie subalgebra $\mathfrak{i}_0$. The subgroup $\tilde{\boldsymbol{I}}_0$ is the connected component of the identity of the inverse image of $\boldsymbol{I}_0$ for the covering map $\tilde{\boldsymbol{G}}_0 \to \boldsymbol{G}_0$, hence closed when $\boldsymbol{I}_0$ is closed. □

EXAMPLE 3.3. – Let $\boldsymbol{G}_0$ be a connected compact Lie group, with a simple Lie algebra $\mathfrak{g}_0$, of rank $\ell \geq 2$. Fix a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ and let $\mathcal{R}$ be the corresponding root system for the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$. Fix a lexicographic order "$\prec$" of $\mathcal{R}$ and let $\alpha_1, \ldots, \alpha_\ell$ be the basis of positive simple roots, and $H_{\alpha_1}, \ldots, H_{\alpha_\ell}$ the corresponding elements of $i\mathfrak{h}_0$. Fix $c_1, \ldots, c_\ell \in \mathbb{R}$, linearly independent over the field $\mathbb{Q}$ of rational numbers. Set $H = c_1 H_{\alpha_1} + \cdots + c_\ell H_{\alpha_\ell}$, and take $\mathfrak{q} = \mathbb{C} \cdot H + \sum_{\alpha \prec 0} \mathfrak{g}^\alpha$, where $\mathfrak{g}^\alpha \subset \mathfrak{g}$ is the root space of $\alpha \in \mathcal{R}$. Then $(\mathfrak{g}_0, \mathfrak{q})$ is fundamental and Levi-nondegenerate, but the analytic subgroup $\boldsymbol{I}_0 = \{\exp(itH) | t \in \mathbb{R}\}$ of $\boldsymbol{G}_0$, corresponding to $\mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0 = i\mathbb{R} \cdot H$, is not closed. Its closure coincides with the Cartan subgroup $\boldsymbol{H}_0$ of $\boldsymbol{G}_0$ with Lie algebra $\mathfrak{h}_0$. We note that its $\boldsymbol{G}_0$-closure (see § 3.2) is $\mathfrak{q}^{\boldsymbol{G}_0} = \mathfrak{q} + \mathbb{C}\mathfrak{h}_0$. It is a Borel subalgebra of $\mathfrak{g}$, and $(\mathfrak{g}_0, \mathfrak{q}^{\boldsymbol{G}_0})$ is the totally complex *CR* algebra of a complex flag manifold.

3.2 – *The* $\boldsymbol{G}_0$*-closure of a CR algebra.*

We may canonically associate to every *CR* algebra a *factual CR* algebra.

PROPOSITION 3.4. – *Let* $\boldsymbol{G}_0$ *be a Lie group with Lie algebra* $\mathfrak{g}_0$*, and* $(\mathfrak{g}_0, \mathfrak{q})$ *a CR algebra. Let* $\boldsymbol{I}_0^0 \subset \boldsymbol{G}_0$ *be the analytic subgroup of* $\mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0$. *Denote by* $\bar{\boldsymbol{I}}_0^0$ *the closure of* $\boldsymbol{I}_0^0$ *in* $\boldsymbol{G}_0$*, by* $\mathfrak{i}^{\boldsymbol{G}_0} \subset \mathfrak{g}_0$ *the Lie algebra of* $\bar{\boldsymbol{I}}_0^0$ *and by* $\mathfrak{i}^{\boldsymbol{G}_0}$ *its complexification. Then:*

(1) $\mathfrak{q}^{\boldsymbol{G}_0} = \mathfrak{q} + \mathfrak{i}^{\boldsymbol{G}_0}$ *is a complex Lie subalgebra of the complexification* $\mathfrak{g}$ *of* $\mathfrak{g}_0$*, which contains* $\mathfrak{q}$ *as an ideal. The quotient* $\mathfrak{q}^{\boldsymbol{G}_0}/\mathfrak{q}$ *is Abelian.*

(2) *If* $\mathfrak{i}_0'$ *is any real linear subspace of* $\mathfrak{g}_0$ *with* $\mathfrak{i}_0 \subset \mathfrak{i}_0' \subset \mathfrak{i}_0^{\boldsymbol{G}_0}$, *and* $\mathfrak{i}'$ *its complexification, then* $\mathfrak{q}' = \mathfrak{q} + \mathfrak{i}'$ *is a complex Lie subalgebra of* $\mathfrak{g}$ *and the* $\mathfrak{g}_0$*-equivariant map* $(\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0, \mathfrak{q}')$ *is an algebraic CR submersion, with Levi-flat fibers.*

(3) *If* $(\mathfrak{g}_0, \mathfrak{q})$ *is fundamental, and* $\mathfrak{q}'$ *is as in* (2), *then also* $(\mathfrak{g}_0, \mathfrak{q}')$ *is fundamental.*

PROOF. – Since $\mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$ for all $g \in \boldsymbol{I}_0^0$, we also have $\mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}$ for all $g \in \bar{\boldsymbol{I}}_0^0$. This implies that $\mathrm{ad}(X)(\mathfrak{q}) \subset \mathfrak{q}$ for all $X \in \mathfrak{i}_0^{\boldsymbol{G}_0}$, hence $\mathfrak{q}^{\boldsymbol{G}_0} = \mathfrak{q} + \mathfrak{i}^{\boldsymbol{G}_0}$ is a complex Lie subalgebra. Clearly $\mathfrak{q}$ is an ideal in $\mathfrak{q}^{\boldsymbol{G}_0}$ and $\mathfrak{q}^{\boldsymbol{G}_0}/\mathfrak{q}$ is Abelian. Indeed the equality $[\mathfrak{i}_0^{\boldsymbol{G}_0}, \mathfrak{i}_0^{\boldsymbol{G}}] = [\mathfrak{i}_0, \mathfrak{i}_0] \subset \mathfrak{i}_0$ (see e.g. [30, Chap. 2 § 5.2]) implies that $[\mathfrak{q}^{\boldsymbol{G}_0}, \mathfrak{q}^{\boldsymbol{G}_0}] = \mathfrak{q}$. Hence, if $\mathfrak{i}_0 \subset \mathfrak{i}_0' \subset \mathfrak{i}_0^{\boldsymbol{G}_0}$, we obtain $[\mathfrak{i}_0', \mathfrak{i}_0'] \subset \mathfrak{i}_0$. Therefore $\mathfrak{q}'$ defined in (2) is a complex Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$, and the $\mathfrak{g}$-equivariant map $(\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0, \mathfrak{q}')$ is an algebraic *CR* submersion, with Levi-flat fibers. Finally, (3) is obvious from the inclusion $\mathfrak{q} \subset \mathfrak{q}'$. □

Keeping the notation of Proposition 3.4, we give the following:

DEFINITION 3.5. – *Let* $(\mathfrak{g}_0, \mathfrak{q})$ *be a CR algebra and* $\boldsymbol{G}_0$ *a Lie group with Lie algebra* $\mathfrak{g}_0$. *The CR algebra* $(\mathfrak{g}_0, \mathfrak{q}^{\boldsymbol{G}_0})$ *is called the* $\boldsymbol{G}_0$*-closure of* $(\mathfrak{g}_0, \mathfrak{q})$ *(cf.* [30, pp. 53-54], *where* $\mathfrak{i}_0^{\boldsymbol{G}_0}$ *is called* the Malcev-closure of $\mathfrak{i}_0$*).*

PROPOSITION 3.6. – *Let* $(\mathfrak{g}_0, \mathfrak{q})$ *be a weakly nondegenerate CR algebra,* $\mathfrak{i}_0'$ *any real linear subspace of* $\mathfrak{g}_0$ *with* $\mathfrak{i}_0 \subset \mathfrak{i}_0' \subset \mathfrak{i}_0^{\boldsymbol{G}_0}$, *and denote by* $\mathfrak{i}'$ *its complexification. Set* $\mathfrak{q}' = \mathfrak{q} + \mathfrak{i}'$. *Then the* $\mathfrak{g}_0$*-equivariant algebraic-CR submersion* $(\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0, \mathfrak{q}')$ *has totally real fibers.*

PROOF. – Indeed, the intersection $\mathfrak{a}_0 = \mathfrak{i}_0' \cap (\mathfrak{q} + \bar{\mathfrak{q}})$ is a real Lie subalgebra that normalizes $\mathfrak{q}$, hence $\mathfrak{q}'' = \mathfrak{q} + \mathfrak{a}$, where $\mathfrak{a}$ is the complexification of $\mathfrak{a}_0$, is a complex Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{q} \subset \mathfrak{q}'' \subset \mathfrak{q} + \bar{\mathfrak{q}}$, the assumption that $(\mathfrak{g}, \mathfrak{q})$ is weakly nondegenerate implies that $\mathfrak{q}'' = \mathfrak{q}$. Then $\mathfrak{a}_0 \subset \mathfrak{q}$ and $\mathfrak{a}_0 \subset \mathfrak{i}_0$, and the fiber of the $\mathfrak{g}_0$-equivariant *CR* map $(\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0, \mathfrak{q}')$ is $(\mathfrak{i}_0', \mathfrak{a} \cap \mathfrak{q}) = (\mathfrak{i}_0', \mathfrak{a})$, thus totally real. □

4. – Real analytic and algebraic *CR* manifolds.

4.1 – *CR submanifolds of analytic spaces.*

A $\boldsymbol{G}_0$-homogeneous *CR* manifold is real analytic, hence it can be realized as a generic *CR* submanifold of a complex manifold (see e.g. [3, 31]). Let us consider, in general, the embedding of a real analytic *CR* manifold into a complex space.

DEFINITION 4.1. – *Let M be a real analytic CR manifold, N a complex space, and $\phi : M \hookrightarrow N$ a real analytic map. The structure sheaf $\mathcal{O}_N$ of germs of holomorphic functions on N pulls back to a subsheaf $\phi^*(\mathcal{O}_N)$ of the sheaf $\mathcal{A}_M$ of germs of complex valued real analytic functions on M. Let $\mathcal{O}_M$ be the sheaf of germs of real analytic CR functions on M. We say that ϕ is*

(1) *a CR map if $\phi^*(\mathcal{O}_N)$ is contained in $\mathcal{O}_M$,*
(2) *a CR immersion if $\phi^*(\mathcal{O}_N) = \mathcal{O}_M$,*
(3) *a generic CR immersion if moreover the composition*

$$\phi^{-1}(\mathcal{O}_N) \longrightarrow \phi^*(\mathcal{O}_N) \longrightarrow \mathcal{O}_M$$

defines an isomorphism of the inverse image sheaf $\phi^{-1}(\mathcal{O}_N)$ onto $\mathcal{O}_M$.

A CR immersion ϕ that is also a topological embedding will be called a CR embedding.

If N is a smooth complex manifold, these notions coincide with the classic definitions in § 1.1.

LEMMA 4.2. – *Let M be a real analytic CR manifold, generically embedded into a complex space N. Then M is contained in the set N^{reg} of non singular points of N.*

PROOF. – Indeed, the fact that M is real analytic implies that for each $p \in M$ the local ring $\mathcal{O}_{M,p}$ is regular. Hence $\mathcal{O}_{N,p}$, being isomorphic to a regular local ring, is also regular. □

4.2 – *Algebraic and weakly-algebraic CR manifolds.*

We consider now *CR* structures on real algebraic manifolds.

DEFINITION 4.3. – *An* affine *CR* submanifold *of $\mathbb{C}^n$ is a smooth real algebraic subvariety M of $\mathbb{C}^n$ that is also a CR submanifold.*

An affine *CR* manifold *is a smooth real algebraic variety, endowed with a CR structure, and CR-isomorphic, by a smooth birational correspondence, with an affine CR submanifold of some $\mathbb{C}^n$.*

An algebraic *CR* manifold *is a smooth real algebraic variety, endowed with a CR structure, in which each point has a Zariski open neighborhood that is an affine CR manifold.*

An algebraic *CR* submanifold *M of an algebraic complex variety N is a smooth real algebraic subvariety of N, embedded as a CR submanifold in the set N^{reg} of its regular points.*

Likewise, we can define semialgebraic *CR manifolds and submanifolds.*

A weakly-algebraic *CR* manifold *M is a smooth real algebraic variety endowed with an algebraic formally integrable partial complex structure. This is given by a formally integrable smooth complex valued algebraic distribution* $T^{0,1}M \subset T^{\mathbb{C}}M$, *with* $T^{0,1}M \cap \overline{T^{0,1}M} = 0_M$.

We observe that an irreducible real algebraic subvariety M' of an irreducible complex algebraic variety N contains a maximal Zariski open subset M that is a real algebraic *CR* submanifold of a Zariski open subset of N.

An algebraic (respectively, semialgebraic) *CR* submanifold of a complex algebraic variety naturally is an algebraic (respectively, semialgebraic) *CR* manifold.

REMARK 4.4. – Since neither the complex nor the real Frobenius theorems are valid in the algebraic category, weakly-algebraic *CR* manifolds may not be algebraic *CR* manifolds. For instance, consider the complex structure on $\mathbb{R}^2_{x,y}$ defined by $J_{x,y} = \begin{pmatrix} x & 1+x^2 \\ -1 & -x \end{pmatrix}$. This structure is weakly-algebraic, but not algebraic, because any rational function in $\mathbb{C}(x,y)$, holomorphic for this structure near a point of $\mathbb{R}^2$, is constant.

PROPOSITION 4.5. – *Let M be an algebraic CR manifold. Then M has a real algebraic embedding into a complex variety N, that is also a generic CR-embedding into the set* N^{reg} *of its regular points.*

PROOF. – We first consider the case where M is an affine *CR* manifold, of *CR* dimension n, and *CR* codimension k, of a Euclidean complex space $\mathbb{C}^{\ell}$. Denote by $\mathcal{I}$ the ideal of polynomials $P \in \mathbb{C}[z_1, \dots, z_\ell]$ vanishing on M. We claim that $N = V(\mathcal{I})$ has the properties requested in the statement. To this aim, let us fix a point $z^0 \in M$. We can assume that the restrictions to M of $dz_1, \dots, dz_{n+k}$ are linearly independent in a neighborhood of z^0 in M, and that $\operatorname{Re} z_1, \dots, \operatorname{Re} z_{n+k}$ and $\operatorname{Im} z_1, \dots, \operatorname{Im} z_n$ define a system of coordinates in a neighborhood of z_0 for the real analytic structure of M. The restrictions to M of the polynomials in $P \in \mathbb{C}[z_1, \dots, z_\ell, \bar{z}_1, \dots, \bar{z}_\ell]$ form a ring which is an algebraic extension of the ring of the restrictions to M of polynomials in $\mathbb{C}[z_1, \dots, z_{n+k}, \bar{z}_1, \dots, \bar{z}_n]$. Let $w \in \mathbb{C}[z_1, \dots, z_\ell]$. Then there is a smallest integer $d \geq 1$ such that w satisfies a monic equation:

$$w^d + a_1(z', \bar{z}')w^{d-1} + \cdots + a_d(z', \bar{z}') = 0 \quad \text{on } M,$$

where $a_j(z', \bar{z}')$ are rational functions of $z_1, \dots, z_{n+k}, \bar{z}_1, \dots, \bar{z}_n$, for $j = 1, \dots, d$. Eliminating denominators, we obtain an equation:

$$b_0(z', \bar{z}')w^d + b_1(z', \bar{z}')w^{d-1} + \cdots + b_d(z', \bar{z}') = 0 \quad \text{on } M, \tag{4.1}$$

with b_j polynomials in $\mathbb{C}[z_1, \ldots, z_{n+k}, \bar{z}_1, \ldots, \bar{z}_n]$. Moreover, we can assume that $b_0 \in \mathbb{C}[z_1, \ldots, z_{n+k}, \bar{z}_1, \ldots, \bar{z}_n]$ has minimal total degree among the b_0's of the non zero polynomial vectors $(b_0, \ldots, b_d)$ that satisfy (4.1). For $b \in \mathbb{C}[z_1, \ldots, z_{n+k}, \bar{z}_1, \ldots, \bar{z}_n]$ the anti-holomorphic differential $\bar{\partial} b$ is a linear combination of the differentials $d\bar{z}_1, \ldots, d\bar{z}_n$ and me may identify $\bar{\partial}_M b$ to the restriction of $\bar{\partial} b$ to M. The pull-backs to M of the differentials $d\bar{z}_1, \ldots,$ $d\bar{z}_n$ are linearly independent on a neighborhood of z_0. Taking $\bar{\partial}_M$ of both sides of (4.1), we obtain:

$$w^d \bar{\partial} b_0 + w^{d-1} \bar{\partial} b_1 + \cdots + w \bar{\partial} b_{d-1} + \bar{\partial} b_d = 0 \quad \text{on } M.$$

By our choice of b_0, this implies that $\bar{\partial} b_0 = 0$ on M, hence that:

$$w^{d-1} \bar{\partial} b_1 + \cdots + \bar{\partial} b_d = 0 \quad \text{on } M.$$

This is a system of polynomial equations with polynomial coefficients on M, thus, by our choice of d, we obtain that all $\bar{\partial} b_j$'s are zero on M, consequently zero because they only depend on $z_1, \ldots, z_{n+k}, \bar{z}_1, \ldots, \bar{z}_{n+k}$. Hence the b_j's are holomorphic, and $a_j = a_j(z') \in \mathbb{C}(z_1, \ldots, z_{n+k})$.

Let $\mathbb{A}$ be the ring of the restrictions to M of the elements of $\mathbb{C}[z_1, \ldots, z_\ell]$, and $\mathbb{B}$ the integral closure in $\mathbb{A}$ of the ring of the restrictions to M of the elements of $\mathbb{C}[z_1, \ldots, z_{n+k}]$. We proved that $\mathbb{A}$ is contained in the integral closure of the field of fractions of $\mathbb{B}$. By the theorem of the primitive element, we can find an element $w \in \mathbb{C}[z_1, \ldots, z_\ell]$, a polynomial $P \in \mathbb{C}[z_1, \ldots, z_{n+k}, w]$, monic with respect to w, such that, if $\Delta(z')$ is the discriminant of P with respect to w:

$$P(z', w) = w^d + a_1(z') w^{d-1} + \cdots + a_d(z') \in \mathcal{I} \tag{4.2}$$

$$\forall j = n+k+1, \ldots, \ell \quad \text{there exists } p_j \in \mathbb{C}[z_1, \ldots, z_{n+k}, w] \text{ such that } \Delta(z') z_j - p_j(z', w) \in \mathcal{I}. \tag{4.3}$$

This shows that $N = V(\mathcal{I})$ is a complex algebraic subvariety of $\mathbb{C}^\ell$, of pure dimension $n + k$. The statement follows, because the points of M are contained in N^{reg} by Lemma 4.2.

The proof in the general case is obtained by patching together a finite atlas of affine charts of M by birational equivalence. □

4.3 – *Homogeneous algebraic CR manifolds.*

Let $\mathfrak{g}_0$ be a finite dimensional real Lie algebra and $\mathfrak{g}$ its complexification. We recall that $\mathfrak{g}_0$ (and $\mathfrak{g}$) are *algebraic Lie algebras* if any of the three equivalent conditions below is satisfied (see [10] for the definition of *replica*, [11], [16] for the equivalence of the conditions):

(1) there exists a real linear algebraic group $\boldsymbol{G}_0$ with Lie algebra $\mathfrak{g}_0$;
(2) there exists an algebraic subgroup of $\mathrm{Aut}(\mathfrak{g}_0)$ with Lie algebra $\mathrm{ad}(\mathfrak{g}_0)$;
(3) for every X in $\mathfrak{g}_0$, the subalgebra $\mathrm{ad}(\mathfrak{g}_0)$ of $\mathfrak{gl}_{\mathbb{R}}(\mathfrak{g}_0)$ contains all replicas of $\mathrm{ad}(X)$.

Moreover, $\mathfrak{g}_0$ is a real algebraic Lie algebra if and only if its complexification $\mathfrak{g}$ is a complex algebraic Lie algebra, and the characterization of complex algebraic Lie algebras is given by conditions that are completely analogous to the ones listed above.

When $\mathfrak{g}_0$ is an algebraic Lie subalgebra of some $\mathfrak{gl}(n, \mathbb{R})$, the semisimple and nilpotent components X_s and X_n of an element X of $\mathfrak{g}_0$ are replicas of X. Thus, in particular, an algebraic Lie algebra $\mathfrak{g}_0$ is ad-splittable: this means that, for every $X \in \mathfrak{g}_0$, also $X_s, X_n \in \mathfrak{g}_0$. Moreover, $\mathrm{ad}(X_s) = [\mathrm{ad}(X)]_s$ and $\mathrm{ad}(X_n) = [\mathrm{ad}(X)]_n$ are the semisimple and the nilpotent components of $\mathrm{ad}(X)$, respectively.

LEMMA 4.6. – *Let $\boldsymbol{G}_0$ be a real linear algebraic group. If M is a $\boldsymbol{G}_0$-homogeneous real algebraic manifold and a smooth $\boldsymbol{G}_0$-homogeneous CR-manifold, then M is a weakly-algebraic CR manifold.*

PROOF. – Fix $p_0 \in M$ and let $(\mathfrak{g}_0, \mathfrak{q})$ be the *CR* algebra of M at p_0. The complexification $T^{\mathbb{C}}\boldsymbol{G}_0$ of the tangent space of $\boldsymbol{G}_0$ is algebraic and can be identified with the Cartesian product $\boldsymbol{G}_0 \times \mathfrak{g}$, the left action of $\boldsymbol{G}_0$ on $\boldsymbol{G}_0 \times \mathfrak{g}$ being defined by:

$$h \cdot (g, Z) = (hg, \mathrm{Ad}_{\mathfrak{g}}(h)(Z)) = (h \circ g, h \circ Z \circ h^{-1}).$$

The set

$$\mathfrak{T} = \{(g, Z) \in \boldsymbol{G}_0 \times \mathfrak{g} \mid g^{-1} \circ Z \circ g \in \mathfrak{q}\}$$

is algebraic. The set $T^{0,1}M$ is the image of $\mathfrak{T}$ by the differential of the map $\boldsymbol{G}_0 \ni g \to g \cdot p_0 \in M$, hence algebraic. This proves that the $\boldsymbol{G}_0$-homogeneous *CR* structure of M defined by $(\mathfrak{g}_0, \mathfrak{q})$ is weakly-algebraic. □

From Lemma 4.6 we obtain:

THEOREM 4.7. – *Let $\mathfrak{g}_0$ be an algebraic Lie algebra. A necessary and sufficient condition for the existence of a homogeneous weakly-algebraic CR manifold M with CR algebra $(\mathfrak{g}_0, \mathfrak{q})$ is that*

(4.4) $\quad \forall X \in \mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0$ *all replicas of* $\mathrm{ad}(X)$ *belong to* $\mathrm{ad}(\mathfrak{i}_0)$.

PROOF. – Let $\mathfrak{z}_0$ be the center of $\mathfrak{g}_0$. By taking the quotient by the central ideal $\mathfrak{i}_0 \cap \mathfrak{z}_0$, we reduce to the case where $\mathfrak{z}_0 \cap \mathfrak{i}_0 = 0$. Then we decompose $\mathfrak{i}_0$ into the direct sum of an $\mathrm{ad}_{\mathfrak{g}_0}$-reductive subalgebra $\mathfrak{l}_0$ and an ideal $\mathfrak{n}_0$ consisting of $\mathrm{ad}_{\mathfrak{g}_0}$-nilpotent elements. We can consider a maximal reductive subalgebra $\mathfrak{l}_0^*$ of $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{g}_0)$ containing $\mathrm{ad}_{\mathfrak{g}_0}(\mathfrak{l}_0)$, and construct, as in [18, XVIII.1], an embedding of $\mathfrak{g}_0$ as an algebraic subalgebra:

$$\phi : \mathfrak{g} \to \mathfrak{l}^* \rtimes \mathfrak{n}_0 \subset \mathfrak{gl}(n, \mathbb{R}), \tag{4.5}$$

where $\mathfrak{n}_0$ is the maximum nilpotent ideal of $\mathfrak{g}_0$, for which the corresponding connected and simply connected Lie group has the structure of an algebraic group, consisting of unipotent matrices. Since, by [17], $\phi(X)$ is a nilpotent matrix in $\mathfrak{gl}(n, \mathbb{R})$ for all $\mathrm{ad}_{\mathfrak{g}_0}$-nilpotent X in $\mathfrak{g}_0$, we obtain that the semisimple parts in $\mathfrak{gl}(n, \mathbb{R})$ of the elements of $\phi(\mathfrak{l}_0)$ belong to $\phi(\mathfrak{l}_0)$, and $\phi(\mathfrak{l}_0)$ is an algebraic subalgebra of $\phi(\mathfrak{g}_0)$ by [11]. Finally, $\phi(\mathfrak{n}_0)$ is algebraic because it is a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ consisting of nilpotent matrices. Hence $\phi(\mathfrak{g}_0) = \phi(\mathfrak{l}_0) \rtimes \phi(\mathfrak{n}_0)$ is an algebraic subalgebra of $\mathfrak{gl}(n, \mathbb{R})$, and the statement follows from Lemma 4.6. □

PROPOSITION 4.8. – *Let $\mathfrak{g}_0$ be an algebraic real Lie algebra and $\mathfrak{q}$ an ideal of its complexification $\mathfrak{g}$, defining a complex structure on $\mathfrak{g}_0$. This means that:*

$$\mathfrak{g} = \mathfrak{q} \oplus \bar{\mathfrak{q}} \quad (\textit{direct sum of ideals}). \tag{4.6}$$

Then we can find a complex algebraic group $\boldsymbol{G}_0$ with associated CR algebra $(\mathfrak{g}_0, \mathfrak{q})$.

PROOF. – We prove that $\mathfrak{q}$ is an algebraic Lie subalgebra of $\mathfrak{g}$. To this aim we observe that $[\mathfrak{q}, \bar{\mathfrak{q}}] = 0$. Hence the centralizer of $\bar{\mathfrak{q}}$ in $\mathfrak{g}^{\mathbb{C}}$ is:

$$\mathfrak{z}_{\mathfrak{g}}(\bar{\mathfrak{q}}) = \mathfrak{q} + \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}).$$

This is an algebraic Lie subalgebra of $\mathfrak{g}$ and therefore $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{q}) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{z}_{\mathfrak{g}}(\bar{\mathfrak{q}}))$ is algebraic in $\mathfrak{gl}_{\mathbb{C}}(\mathfrak{g})$. By the same argument of Theorem 4.7, we obtain that there exists a complex algebraic group $\boldsymbol{G}$ and a complex algebraic normal subgroup $\boldsymbol{Q}$ with Lie algebra $\mathfrak{q}$. Then $\boldsymbol{G}_0 = \boldsymbol{G}/\boldsymbol{Q}$ is a complex algebraic group satisfying the conditions of the statement. □

EXAMPLE 4.9. – Let $\mathfrak{g}_0 = \mathfrak{sl}(2m, \mathbb{R})$, with $m \geq 2$. Consider a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g} \simeq \mathfrak{sl}(2m, \mathbb{C})$ such that $\mathfrak{h} = \mathfrak{b} \cap \bar{\mathfrak{b}}$ is a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is the complexification of a maximally compact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilpotent ideal of $\mathfrak{b}$. Fix an element $H \in \mathfrak{h} \setminus \mathfrak{h}_0$, such that $\exp(\mathbb{R} H)$ is not closed in $\boldsymbol{SL}(2m, \mathbb{C})$. We choose $\mathfrak{q} = \mathfrak{n} \oplus \mathbb{C} H$. Since $\mathfrak{q} \cap \bar{\mathfrak{q}} = 0$, the complex Lie subalgebra $\mathfrak{q}$ defines a left homogeneous *CR* structure on $\boldsymbol{SL}(2m, \mathbb{R})$. However, in this case there is no semialgebraic $\boldsymbol{G}_0$-equivariant *CR* embedding of $\boldsymbol{G}_0 \simeq \boldsymbol{SL}(2m, \mathbb{R})$ into an $\boldsymbol{SL}(2m, \mathbb{C})$-homogeneous complex manifold.

EXAMPLE 4.10. – Let $\mathfrak{g} = \mathfrak{sl}(2m-1, \mathbb{C})$, with $m \geq 2$. Define the conjugation:

$$A = (a_{i,j})_{1 \leq i,j \leq 2m-1} \to \bar{A} = (\bar{a}_{2m-i,2m-j})_{1 \leq i,j \leq 2m-1}.$$

Consider the real form $\mathfrak{g}_0 = \{A = \bar{A}\} \simeq \mathfrak{sl}(2m-1,\mathbb{R})$ of $\mathfrak{g}$, and let $\boldsymbol{G}_0 \simeq \boldsymbol{SL}(2m-1,\mathbb{R})$ be the analytic Lie subgroup of $\boldsymbol{SL}(n,\mathbb{C})$ with Lie algebra $\mathfrak{g}_0$. Define:

$$\mathfrak{q} = \{A = (a_{i,j}) \in \mathfrak{sl}(2m-1,\mathbb{C}) \mid a_{i,j} = 0 \text{ for } i > j \text{ and for } i = j > m\}.$$

Since $\mathfrak{q} \cap \bar{\mathfrak{q}} = 0$ and $\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{sl}(2m-1,\mathbb{C})$, the datum of the *CR* algebra $(\mathfrak{g},\mathfrak{q})$ yields on $\boldsymbol{G}_0$ a complex structure, which is only left $\boldsymbol{G}_0$-invariant. Note that, being $\boldsymbol{SL}(2m-1,\mathbb{R})$ a simple real Lie group corresponding to a connected Satake diagram, it cannot carry the structure of either a complex Lie group, or a complex algebraic group. However, it is a quasi-projective smooth complex variety, open in $\boldsymbol{SL}(2m-1,\mathbb{C})/\boldsymbol{Q}$, where $\boldsymbol{Q}$ is the algebraic subgroup of $\boldsymbol{SL}(2m-1,\mathbb{C})$ corresponding to the solvable Lie subalgebra $\mathfrak{q}$.

EXAMPLE 4.11. – Let us take $\mathfrak{g}_0$ and $\boldsymbol{G}_0$ as in Example 4.10, with $m = 2$, and define:

$$\mathfrak{q} = \left\{ A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ 0 & a_{2,2} & a_{2,3} \\ 0 & 0 & a_{3,3} \end{pmatrix} \in \mathfrak{sl}(3,\mathbb{C}) \;\middle|\; a_{1,1} + \lambda\, a_{3,3} = 0 \right\},$$

for an irrational complex number λ, with $|\lambda| \neq 1$. We have $\mathfrak{q} \cap \bar{\mathfrak{q}} = 0$ and $\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{sl}(3,\mathbb{C})$. Since the subalgebra $\mathfrak{q}$ is not algebraic, the complex structure on $\boldsymbol{G}_0$ defined by $(\mathfrak{g}_0,\mathfrak{q})$ is weakly algebraic, but not algebraic.

THEOREM 4.12. – *Let $\boldsymbol{G}_0$ be a real linear algebraic group. Let M be a real algebraic manifold and a smooth CR manifold, on which $\boldsymbol{G}_0$ acts as a transitive group of algebraic and CR transformations. Let $(\mathfrak{g}_0,\mathfrak{q})$ be the CR algebra of M at a point p_0. Then*:

(1) $\mathfrak{q}$ *is an algebraic subalgebra of* $\mathfrak{g}$.

(2) *There are a $\boldsymbol{G}$-homogeneous complex algebraic manifold $\hat{M}$ and a $\boldsymbol{G}_0$-equivariant CR generic algebraic embedding $M \hookrightarrow \hat{M}$.*

Vice versa, if $\mathfrak{q}$ is algebraic, then M is a $\boldsymbol{G}_0$-homogeneous algebraic CR manifold.

PROOF. – First assume that M is affine. By Proposition 4.5, there is a generic *CR*-embedding $M \hookrightarrow N$ of M into the set of regular points of an affine complex variety $N \hookrightarrow \mathbb{C}^{\ell}$, in such a way that the ring $\mathbb{C}[M]$ of regular *CR* functions on M coincides with the ring $\mathbb{C}[N]$ of regular holomorphic functions on N.

Analogously, $\mathbb{C}[\boldsymbol{G}_0] = \mathbb{C}[\boldsymbol{G}]$. Let $\boldsymbol{I}_0 \subset \boldsymbol{G}_0$ be the isotropy subgroup at $p_0 \in M$ and $\pi : \boldsymbol{G}_0 \to \boldsymbol{G}_0/\boldsymbol{I}_0 = M$ the natural projection. The subring $\pi^*(\mathbb{C}[M])$ of $\mathbb{C}[\boldsymbol{G}_0] = \mathbb{C}[\boldsymbol{G}]$ is $\boldsymbol{G}_0$-invariant, hence also $\boldsymbol{G}$-invariant. Thus it defines a $\boldsymbol{G}$-

homogeneous complex algebraic variety $\hat{M}$, and the isotropy subgroup is an algebraic subgroup $\boldsymbol{Q}$ with Lie algebra $\mathfrak{q}$. Indeed, in a $\boldsymbol{G}_0$-equivariant way we have $\mathbb{C}[M] = \mathbb{C}[\hat{M}]$, and we also obtain a generic algebraic *CR* embedding $M \hookrightarrow \hat{M}$.

Let us now turn to the general case. Let $M \hookrightarrow N^{\text{reg}}$ be the embedding of Proposition 4.5, and let $\mathcal{R}_M$ and $\mathcal{R}_N$ be the sheaves of germs of regular *CR* functions on M and of regular holomorphic functions on N, respectively. Then $\mathcal{R}_M$ and $\mathcal{R}_N$ are isomorphic, as, for every open $U \subset N$, the restriction map $\mathcal{R}_N(U) \to \mathcal{R}_M(U \cap M)$ is an isomorphism. Then we apply the considerations of the affine case to the subsheaf $\pi^{-1}(\mathcal{R}_M)$ of $\mathcal{R}_{\boldsymbol{G}_0} \simeq \mathcal{R}_{\boldsymbol{G}}$.

When $\mathfrak{q}$ is algebraic, we consider the algebraic subgroup $\boldsymbol{Q}$ of $\boldsymbol{G}$ with Lie algebra $\mathfrak{q}$ and the natural $\boldsymbol{G}_0$-equivariant embedding $M \hookrightarrow \boldsymbol{G}/\boldsymbol{Q}$. □

4.4 – *Algebraic closure of a CR algebra.*

The considerations of § 3.2 can be adapted to the case of a real linear algebraic group $\boldsymbol{G}_0$. Indeed, if $\boldsymbol{H}_0$ is any subgroup of $\boldsymbol{G}_0$, we can define its *algebraic* closure $\boldsymbol{H}_0^{\text{alg}}$ as the smallest algebraic subgroup of $\boldsymbol{G}_0$ containing $\boldsymbol{H}_0$. It coincides with the closure of $\boldsymbol{H}_0$ in the Zariski topology of $\boldsymbol{G}_0$. When $\boldsymbol{H}_0$ is the analytic subgroup of $\boldsymbol{G}_0$ corresponding to a Lie subalgebra $\mathfrak{h}_0$ of its Lie algebra $\mathfrak{g}_0$, we denote by $\mathfrak{h}_0^{\text{alg}}$ the Lie algebra of $\boldsymbol{H}_0^{\text{alg}}$. Also in this case we have (see [33, Theorem 6.2])

$$[\mathfrak{h}_0, \mathfrak{h}_0] = [\mathfrak{h}_0^{\text{alg}}, \mathfrak{h}_0^{\text{alg}}]. \tag{4.7}$$

As in § 3.2, we have:

PROPOSITION 4.13. – *Let $\boldsymbol{G}_0$ be a real linear algebraic group, with Lie algebra $\mathfrak{g}_0$. Let $(\mathfrak{g}_0, \mathfrak{q})$ be a CR algebra, and denote by $\mathfrak{i}_0^{\text{alg}}$ the algebraic closure of the isotropy subalgebra $\mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0$. Set $\mathfrak{q}^{\text{alg}} = \mathfrak{q} + \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{i}_0^{\text{alg}}$. Then $\mathfrak{q}^{\text{alg}}$ is a complex Lie subalgebra of $\mathfrak{g}$, contained in the normalizer of $\mathfrak{q}$ in $\mathfrak{g}$. The quotient $\mathfrak{q}^{\text{alg}}/\mathfrak{q}$ is Abelian.*

Fix any real linear subspace $\mathfrak{i}'_0$ of $\mathfrak{g}_0$ with $\mathfrak{i}_0 \subset \mathfrak{i}'_0 \subset \mathfrak{i}_0^{\text{alg}}$ and define $\mathfrak{q}' = \mathfrak{q} + \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{i}'_0$. Then:

(1) *$\mathfrak{q}'$ is a complex Lie subalgebra of $\mathfrak{g}$ and the $\mathfrak{g}_0$-equivariant map $(\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0, \mathfrak{q}')$ is an algebraic CR submersion, with Levi-flat fibers.*

(2) *If $(\mathfrak{g}_0, \mathfrak{q})$ is fundamental, then also $(\mathfrak{g}_0, \mathfrak{q}')$ is fundamental.*

(3) *If $(\mathfrak{g}_0, \mathfrak{q})$ is weakly nondegenerate, then the fiber of the $\mathfrak{g}_0$-equivariant map $(\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0, \mathfrak{q}')$ is totally real.* □

DEFINITION 4.14. – *The CR algebra $(\mathfrak{g}_0, \mathfrak{q}^{\text{alg}})$ is called the* algebraic closure *of the CR algebra $(\mathfrak{g}_0, \mathfrak{q})$.*

4.5 – *The* $\boldsymbol{G}_0$*- and the* $\mathfrak{g}_0$*-anticanonical fibrations.*

In this section we describe the anticanonical fibration of [4] and [15] in terms of *CR* algebras.

Let $\boldsymbol{G}_0$ be a Lie group, $\mathfrak{g}_0$ its Lie algebra. Given a *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$, set:

$$\mathfrak{a}_0 = \boldsymbol{N}_{\mathfrak{g}_0}(\mathfrak{q}) = \{X \in \mathfrak{g}_0 \mid [X, \mathfrak{q}] \subset \mathfrak{q}\} \tag{4.8}$$

$$\mathfrak{q}' = \mathfrak{q} + \mathfrak{a}, \quad \text{with} \quad \mathfrak{a} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a}_0, \tag{4.9}$$

$$\boldsymbol{A}_0 = \boldsymbol{N}_{\boldsymbol{G}_0}(\mathfrak{q}) = \{g \in \boldsymbol{G}_0 \mid \mathrm{Ad}(g)(\mathfrak{q}) = \mathfrak{q}\}. \tag{4.10}$$

Proposition 4.15. – *Keep the notation introduced above. Then:*

(1) $\mathfrak{q}'$ *is the complex Lie subalgebra of* $\mathfrak{g}$ *characterized by the properties:*

$$\begin{cases} \mathfrak{q} \subset \mathfrak{q}', \quad \mathfrak{q}' \cap \mathfrak{g}_0 = \mathfrak{a}_0, \\ (\mathfrak{g}_0, \mathfrak{q}) \to (\mathfrak{g}_0, \mathfrak{q}') \text{ is a } \mathfrak{g}_0\text{-equivariant CR-submersion.} \end{cases} \tag{4.11}$$

(2) $\mathfrak{q}'$ *is the smallest Lie subalgebra of* $\mathfrak{g}$ *satisfying* $\mathfrak{q} + \mathfrak{a}_0 \subset \mathfrak{q}' \subset \boldsymbol{N}_{\mathfrak{g}}(\mathfrak{q})$.

(3) *The fiber* $(\mathfrak{a}_0, \mathfrak{a} \cap \mathfrak{q})$ *of the* $\mathfrak{g}_0$*-equivariant CR fibration* $(\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \mathfrak{q}')$ *is Levi-flat. Indeed* $\mathfrak{a} \cap \mathfrak{q} = \mathfrak{q} \cap \bar{\mathfrak{q}}'$*, and* $[\mathfrak{q} \cap \bar{\mathfrak{q}}', \bar{\mathfrak{q}} \cap \mathfrak{q}'] \subset \mathfrak{q} \cap \bar{\mathfrak{q}}$.

Moreover:

(4) *If* $(\mathfrak{g}_0, \mathfrak{q}')$ *is totally real, then* $(\mathfrak{g}_0, \mathfrak{q})$ *is Levi-flat, and* $[\mathfrak{q}, \bar{\mathfrak{q}}] \subset \mathfrak{q} \cap \bar{\mathfrak{q}}$.

(5) *Conditions* (*i*), (*ii*) *and* (*iii*) *below are equivalent and imply* (*iv*):

$$\underbrace{\mathfrak{a}_0 = \mathfrak{g}_0}_{(i)} \iff \underbrace{\mathfrak{q}' = \mathfrak{g}}_{(ii)} \iff \underbrace{\mathfrak{q} \text{ is an ideal of } \mathfrak{g}}_{(iii)} \implies \underbrace{\mathfrak{a}_0 \text{ is an ideal of } \mathfrak{g}_0}_{(iv)}.$$

(6) $\boldsymbol{A}_0$ *is a closed subgroup of* $\boldsymbol{G}_0$ *and hence* $(\mathfrak{g}_0, \mathfrak{q}')$ *is factual.*

(7) *If* $\mathfrak{g}_0$ *is an algebraic Lie algebra, then also* $\mathfrak{a}_0$ *and* $\mathfrak{a}$ *are algebraic. If* $\boldsymbol{G}_0$ *is a real linear algebraic group, then* $M' = \boldsymbol{G}_0/\boldsymbol{A}_0$ *is a weakly algebraic CR manifold. If* $\mathfrak{q}$ *is algebraic, then* $\mathfrak{q}'$ *is algebraic too, and* M' *is an algebraic CR manifold.*

Proof. – Since $\mathfrak{i}_0 = \mathfrak{q} \cap \mathfrak{g}_0 \subset \mathfrak{a}$, by Proposition 1.7 (4.9) and (4.11) are equivalent. This proves (1).

Indeed, any complex Lie subalgebra containing $\mathfrak{a}_0$ contains its complexification $\mathfrak{a}$. Hence (2) is obvious.

By (4.8), we have $[\mathfrak{a}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{a}, \bar{\mathfrak{q}}] \subset \bar{\mathfrak{q}}$, hence $[\mathfrak{a} \cap \mathfrak{q}, \overline{\mathfrak{a} \cap \mathfrak{q}}] = [\mathfrak{a} \cap \mathfrak{q}, \mathfrak{a} \cap \bar{\mathfrak{q}}] \subset \mathfrak{q} \cap \bar{\mathfrak{q}}$ yields (3).

(4). When $(\mathfrak{g}_0, \mathfrak{q}')$ is totally real, $\bar{\mathfrak{q}} \subset \mathfrak{q}'$, hence $[\mathfrak{q}, \bar{\mathfrak{q}}] \subset [\mathfrak{q}, \mathfrak{q}'] \subset \mathfrak{q}$. By conjugation, we obtain $[\mathfrak{q}, \bar{\mathfrak{q}}] \subset \mathfrak{q} \cap \bar{\mathfrak{q}}$.

(5). We clearly have (*i*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*iii*) $\Rightarrow$ (*i*) and (*iii*) $\Rightarrow$ (*iv*).

(6) follows from (4.10), since $\mathfrak{a}_0$ is the Lie algebra of $\boldsymbol{A}_0$, and (7) is a consequence of Theorems 4.7 and 4.12. □

5. – Left invariant *CR* structures on semisimple Lie groups.

In this and the following section, we shall discuss special examples of homogeneous *CR* structures. We begin by investigating left-invariant *CR* structures on real semisimple Lie groups (see e.g. [32]). Note that a Lie group with a left and right invariant complex structure is in fact a complex Lie group.

5.1 – *Existence of maximal CR structures.*

THEOREM 5.1. – *Every semisimple real Lie group of even dimension admits a left invariant complex structure.*

Every semisimple real Lie group of odd dimension admits a left invariant CR structure of hypersurface type.

PROOF. – Let $\boldsymbol{G}_0$ be a simple real Lie group, with Lie algebra $\mathfrak{g}_0$. Take a maximally compact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. The complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ contains a Borel subalgebra $\mathfrak{b}$ with $\mathfrak{b} \cap \bar{\mathfrak{b}}$ equal to the complexification $\mathfrak{h}$ of $\mathfrak{h}_0$ (see e.g. [21]). Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilpotent ideal of $\mathfrak{b}$.

If the dimension of $\mathfrak{g}_0$ is even, then the dimension of $\mathfrak{h}_0$ is even too, and we can find a complex structure $J : \mathfrak{h}_0 \to \mathfrak{h}_0$. Then $V = \{X + iJX \mid X \in \mathfrak{h}_0\}$ is a complex subspace of $\mathfrak{h}$, with $V \cap \bar{V} = \{0\}$. Hence $\mathfrak{q} = V \oplus \mathfrak{n}$ is a complex subalgebra of $\mathfrak{g}$, with $\mathfrak{q} \cap \bar{\mathfrak{q}} = 0$ and $\mathfrak{g} = \mathfrak{q} \oplus \bar{\mathfrak{q}}$. The *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$ defines a left invariant complex structure on $\boldsymbol{G}_0$.

If $\mathfrak{g}_0$ has odd dimension, then $\mathfrak{h}_0$ has odd dimension too. Fix a hyperplane $\mathfrak{m}_0$ of $\mathfrak{h}_0$, a complex structure $J : \mathfrak{m}_0 \to \mathfrak{m}_0$, set $V = \{X + iJX \mid X \in \mathfrak{m}_0\}$ and take $\mathfrak{q} = V \oplus \mathfrak{n}$. Since $\mathfrak{q} \cap \bar{\mathfrak{q}} = 0$ and $\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{n} \oplus \bar{\mathfrak{n}} \oplus \mathfrak{m}$, the *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$ defines a left invariant *CR* structure of hypersurface type on $\boldsymbol{G}_0$. □

EXAMPLE 5.2. – Let $\boldsymbol{G} = \boldsymbol{SL}(n, \mathbb{C})$ and consider on its Lie algebra $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ the conjugation $A \to A^\sharp$, defined by $A^\sharp = (\bar{a}_{n+1-i,n+1-j})_{1 \le i,j \le n}$ for $A = (a_{i,j})_{1 \le i,j \le n}$. Then $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid X^\sharp = X\} \simeq \mathfrak{sl}(n, \mathbb{R})$ and $\boldsymbol{G}_0 = \{g \in \boldsymbol{G} \mid g^\sharp = g\} \simeq \boldsymbol{SL}(n, \mathbb{R})$. The diagonal matrices of $\mathfrak{g}_0$ are a maximally compact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. Let $n = 2m + 1$ be odd. Fix p with $1 < p \le m$ and define $\mathfrak{q}'$ as the complex Lie subalgebra of $\mathfrak{g}$ consisting of matrices $(a_{i,j})_{1 \le i,j \le n}$ with $a_{i,j} = 0$ when either $p < i \le n - p$ and $j \le i$, or $i > n - p$. Let $\mathfrak{a}$ be a subspace of the Cartan subalgebra $\mathfrak{h}$ of the diagonal matrices of $\mathfrak{g}$, with $\lambda_j = \lambda_n$ for $j > n - p$, then $\mathfrak{a} \cap \bar{\mathfrak{a}} = 0$ and $\mathfrak{a} + \bar{\mathfrak{a}} = \mathfrak{h}$. By setting $\mathfrak{q} = \mathfrak{q}' + \mathfrak{a}$, we obtain a *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$, with a non solvable $\mathfrak{q}$, which defines a left invariant complex structure on $\boldsymbol{G}_0 \simeq \boldsymbol{SL}(2m + 1, \mathbb{R})$.

5.2 – *Classification of the regular maximal CR structures.*

We recall that a complex Lie subalgebra $\mathfrak{q}$ of a complex semisimple Lie algebra $\mathfrak{g}$ is *regular* if its normalizer contains a Cartan subalgebra of $\mathfrak{g}$.

DEFINITION 5.3. – *We say that a CR algebra* $(\mathfrak{g}_0, \mathfrak{q})$ *is* regular *if* $\mathfrak{q}$ *is normalized by a Cartan subalgebra of the real Lie algebra* $\mathfrak{g}_0$.

If $\boldsymbol{G}_0$ *is a semisimple real Lie group with Lie algebra* $\mathfrak{g}_0$*, a* $\boldsymbol{G}_0$*-invariant CR structure on a* $\boldsymbol{G}_0$*-homogeneous CR manifold* M *is called* regular *if the associated CR algebra* $(\mathfrak{g}_0, \mathfrak{q})$ *is regular.*

Fix a semisimple real Lie algebra $\mathfrak{g}_0$. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$, and $\mathcal{R}$ the root system of its complexification $\mathfrak{h}$ in $\mathfrak{g}$. For each $\alpha \in \mathcal{R}$ we write $\mathfrak{g}^\alpha$ for the root space of α. The real form $\mathfrak{g}_0$ defines a conjugation in $\mathfrak{g}$, which by duality gives an involution $\alpha \to \bar{\alpha}$ in $\mathcal{R}$, with $\bar{\alpha}(H) = \overline{\alpha(\bar{H})}$ for all $H \in \mathfrak{h}$.

LEMMA 5.4. – *Assume that there is a closed system of roots* $\mathcal{Q} \subset \mathcal{R}$ *with*

$$\mathcal{Q} \cap \bar{\mathcal{Q}} = \emptyset, \quad \mathcal{Q} \cup \bar{\mathcal{Q}} = \mathcal{R}. \tag{5.1}$$

Then $\mathfrak{h}_0$ *is maximally compact.*

Set $\mathcal{Q}^r = \{\alpha \in \mathcal{Q} \mid -\alpha \in \mathcal{Q}\}$ *and* $\mathcal{Q}^n = \{\alpha \in \mathcal{Q} \mid -\alpha \notin \mathcal{Q}\}$. *Then:*

(1) $\mathcal{Q}^r \cup \bar{\mathcal{Q}}$ *and* $\mathcal{Q}^r \cup \bar{\mathcal{Q}}^n$ *are closed systems of roots;*
(2) *the two systems of roots* $\mathcal{Q}^r$ *and* $\bar{\mathcal{Q}}^r$ *are strongly orthogonal;*
(3) $\mathcal{P} = \mathcal{Q} \cup \bar{\mathcal{Q}}^r$ *is parabolic with* $\mathcal{P}^n := \{\alpha \in \mathcal{P} \mid -\alpha \notin \mathcal{P}\} = \mathcal{Q}^n$*;*
(4) *there is a system of simple positive roots* $\alpha_1, \dots, \alpha_\ell$ *of* $\mathcal{R}$ *with the properties:*

$$\begin{cases} \alpha_1, \dots, \alpha_\ell \in \mathcal{P}, \\ \alpha_1, \dots, \alpha_p \quad \textit{is a basis of} \quad \mathcal{Q}^r, \\ \alpha_{p+1}, \dots, \alpha_{\ell-p} \in \mathcal{Q}^n, \\ \bar{\alpha}_i \prec 0 \quad \forall i = 1, \dots, \ell, \\ \bar{\alpha}_i = -\alpha_{\ell+1-i} \quad \textit{for} \quad i = 1, \dots, p. \end{cases} \tag{5.2}$$

PROOF. – By (5.1), $\bar{\alpha} \neq \alpha$ for all $\alpha \in \mathcal{R}$, and this is equivalent to $\mathfrak{h}_0$ being maximally compact (see e.g. [21, Ch. VI, § 6]).

The root system $\mathcal{R}$ is partitioned into minimal disjoint subsets, invariant by addition of roots of $\mathcal{Q}^r$. Since $\mathcal{Q}$ is a union of such $\mathcal{Q}^r$-invariant minimal subsets, its complement $\bar{\mathcal{Q}}$ is $\mathcal{Q}^r$-invariant, too. Likewise, $\mathcal{Q}$ is $\bar{\mathcal{Q}}^r$-invariant. This implies that $\mathcal{Q}^r$ and $\bar{\mathcal{Q}}^r$ are strongly orthogonal. Indeed, assume by contradiction that there are $\alpha, \beta \in \mathcal{Q}^r$ such that $\alpha + \bar{\beta} \in \mathcal{R}$. Then $\alpha + \bar{\beta} \in \mathcal{Q} \cap \bar{\mathcal{Q}}$ would give a contradiction.

Since $\bar{\mathcal{Q}}^r$ is $\mathcal{Q}^r$-invariant, then also $\bar{\mathcal{Q}}^n$ is $\mathcal{Q}^r$-invariant. This proves (1) and (2).

From (5.1) we also deduce that $\bar{\mathcal{Q}}^n$ is equal to $\{-\alpha \mid \alpha \in \mathcal{Q}^n\}$, and this implies (3).

To prove (4), we begin by fixing an element $A_0 \in \mathfrak{h}_{\mathbb{R}}$ that defines the parabolic set $\mathcal{P}$:

$$\mathcal{P} = \{\alpha \in \mathcal{R} \mid \alpha(A_0) \geq 0\}.$$

Next we note that, since $\mathcal{R}$ does not contain any real root, there is a regular element A_1 in $\mathfrak{h}_{\mathbb{R}}$ with $\bar{A}_1 = -A_1$, i.e. with $iA_1 \in \mathfrak{h}_0$. Take $\varepsilon > 0$ with $|\alpha(A_1)| < \varepsilon^{-1}\alpha(A_0)$ for $\alpha \in \mathcal{Q}^n$. Then $A = A_0 + \varepsilon A_1$ is regular and we shall take $\mathcal{B} = \{\alpha_1, \dots, \alpha_\ell\}$ to be the simple roots of the system of positive roots $\mathcal{R}^+ = \{\alpha \in \mathcal{R} \mid \alpha(A) > 0\}$. Take $\{\alpha_1, \dots, \alpha_p\} = \mathcal{B} \cap \mathcal{Q}^r$ and $\{\alpha_{p+1}, \dots, \alpha_r\} = \mathcal{B} \cap \mathcal{Q}^n$. By our choice of ε and A_1, the set $\{\alpha_1, \dots, \alpha_p\}$ is the set of the simple positive roots in $\{\alpha \in \mathcal{Q}^r \mid \alpha(A_1) > 0\}$. Likewise, the simple roots in $\{\alpha \in \bar{\mathcal{Q}}^r \mid \alpha(A_1) > 0\}$ are contained in $\{\alpha_{\ell-p+1}, \dots, \alpha_\ell\} \subset \mathcal{B}$. Hence $r = \ell - p$.

To conclude the proof of (4), it suffices to note that, since $\overline{\mathcal{R}^+} = \mathcal{R}^- = \{-\alpha \mid \alpha \in \mathcal{R}^+\}$, the conjugate of each simple root is a simple negative root. Thus, by suitably labelling the roots in $\mathcal{B}$, since by (5.1) we have $\bar{\alpha} \neq -\alpha$ for $\alpha \in \mathcal{Q}^r$, we also obtain the last line of (5.2). The proof is complete. □

PROPOSITION 5.5. – *Let $\mathbf{G}_0$ be a real semisimple Lie group. Then any regular CR structure on $\mathbf{G}_0$ of maximal CR dimension is associated with a regular CR algebra $(\mathfrak{g}_0, \mathfrak{q})$, with a $\mathfrak{q}$ that is normalized by a maximally compact Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, and is of the form:*

$$\mathfrak{q} = \mathfrak{m} \oplus \sum_{\alpha \in \mathcal{Q}} \mathfrak{g}^\alpha \tag{5.3}$$

for a closed system of roots $\mathcal{Q} \subset \mathcal{R}$ satisfying (5.1)*, and a complex subspace $\mathfrak{m}$ of the complexification $\mathfrak{h}$ of $\mathfrak{h}_0$, with the properties:*

$$\dim_{\mathbb{C}} \mathfrak{m} = \left[\frac{\ell}{2}\right], \quad \mathfrak{s} \cap \mathfrak{h} \subset \mathfrak{m}, \quad \mathfrak{m} \cap \bar{\mathfrak{m}} = \{0\}. \tag{5.4}$$

Here ℓ is the rank of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ and $\mathfrak{s}$ is the Levi subalgebra of $\mathfrak{q}$ associated with the root system $\mathcal{Q}^r$.

PROOF. – We note that (5.3), with a choice of $\mathcal{Q}$ and $\mathfrak{m}$ satisfying (5.1) and (5.4), has *CR* codimension equal to 0 or 1, according to whether $\mathfrak{g}$ has even or odd rank, respectively. Thus it yields a *CR* structure of maximal *CR* dimension. Let $(\mathfrak{g}_0, \mathfrak{q})$ be a regular *CR* algebra, of codimension at most 1, and set $\mathfrak{m} = \mathfrak{q} \cap \mathfrak{h}$. Then $\mathfrak{m}$ must satisfy (5.4) by the codimension constraint, and the set $\mathcal{Q}$ of the roots α with $\mathfrak{g}^\alpha \subset \mathfrak{q}$ satisfies (5.1). □

EXAMPLE 5.6. – Let $\mathfrak{g} \simeq \mathfrak{sl}(3, \mathbb{R})$, consist of the matrices $A = (a_{i,j}) \in \mathfrak{sl}(3, \mathbb{C})$ that satisfy $\bar{a}_{i,j} = a_{4-i,4-j}$. Set

$$\mathfrak{q} = \left\{ \begin{pmatrix} z_1 & z_2 & 0 \\ z_2 & z_1 & 0 \\ z_3 & z_4 & -2z_1 \end{pmatrix} \middle| \, z_1, z_2, z_3, z_4 \in \mathbb{C} \right\}.$$

The *CR* algebra $(\mathfrak{g}_0, \mathfrak{q})$ defines a left invariant complex structure on $\boldsymbol{G}_0 \simeq \boldsymbol{SL}(3, \mathbb{R})$, because $\mathfrak{q} \cap \bar{\mathfrak{q}} = \{0\}$ and $\mathfrak{q} + \bar{\mathfrak{q}} = \mathfrak{sl}(3, \mathbb{C})$. But $(\mathfrak{g}_0, \mathfrak{q})$ is not regular as a *CR* algebra, since $\mathfrak{q}$ is self-normalizing in $\mathfrak{sl}(3, \mathbb{C})$, hence, in particular, is not normalized by any Cartan subalgebra of $\mathfrak{g}_0$.

In [9, 22] all complex structures on a compact semisimple Lie group of even dimension are shown to be regular. According to the example above, in the cases of non compact semisimple real Lie groups, a complete classification of left invariant maximal *CR* structures would require some extra consideration of non regular structures.

6. – Symmetric *CR* structures on complete flags.

Symmetric maximal *almost-CR* structures (i.e. formally integrability is not required) on *complete flags* were studied in [14]. Here we utilize *CR* algebras to study their *CR*-symmetric (formally integrable) structures, that are also of finite type.

A complete flag is a homogeneous compact complex manifold, which is the quotient $M \simeq \boldsymbol{G}/\boldsymbol{B}$ of a semisimple complex Lie group $\boldsymbol{G}$ by a Borel subgroup $\boldsymbol{B}$. A maximal compact subgroup $\boldsymbol{U}_0$ of $\boldsymbol{G}$ acts transitively on M, which is therefore also a quotient $M \simeq \boldsymbol{U}_0/\boldsymbol{T}_0$ of $\boldsymbol{U}_0$ with respect to a maximal torus $\boldsymbol{T}_0$.

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{u}_0$, $\mathfrak{t}_0$ be the Lie algebras of $\boldsymbol{G}$, $\boldsymbol{B}$, $\boldsymbol{U}_0$, $\boldsymbol{T}_0$, respectively. Then $\mathfrak{g}$ is complex semisimple and is the complexification of its compact form $\mathfrak{u}_0$. The complexification $\mathfrak{h}$ of $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}$, contained in $\mathfrak{b}$.

6.1 – *Homogeneous CR structures on complete flags.*

We shall consider M as a *real* compact manifold, and discuss its $\boldsymbol{U}_0$-homogeneous *CR* structures. By Proposition 1.3, having fixed the point $\mathfrak{o} = [\boldsymbol{T}_0]$ of M, the $\boldsymbol{U}_0$-homogeneous *CR* structures on M are in one-to-one correspondence with the complex Lie subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ satisfying $\mathfrak{q} \cap \mathfrak{u}_0 = \mathfrak{t}_0$. In particular, any such $\mathfrak{q}$ contains the Cartan subalgebra $\mathfrak{h}$, hence is *regular*. Denote by $\mathcal{R}$ the root system of $\mathfrak{h}$ in $\mathfrak{g}$, and let $\mathcal{Q}$ be the subset of $\mathcal{R}$ consisting of the roots α for which $\mathfrak{g}^{\alpha} \subset \mathfrak{q}$. Then

$$\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{n}, \quad \text{where} \quad \mathfrak{n} = \sum_{\alpha \in \mathcal{Q}} \mathfrak{g}^{\alpha}. \tag{6.1}$$

Conjugation with respect to the real form $\mathfrak{u}_0$ yields on $\mathcal{R}$ the involution $\alpha \to \bar{\alpha} = -\alpha$. Thus, the assumption that $\mathfrak{q} \cap \bar{\mathfrak{q}} = \mathfrak{h}$ is equivalent to $\mathcal{Q} \cap (-\mathcal{Q}) = \emptyset$. Hence $\mathfrak{q}$ is solvable (see e.g. [33, Proposition 1.2, p. 183]), and

$$\mathfrak{h} \subset \mathfrak{q} \subset \mathfrak{b}. \tag{6.2}$$

We may consider the ordering of $\mathcal{R}$ for which the roots α with $\mathfrak{g}^\alpha \subset \mathfrak{b}$ are positive, so that $\mathcal{Q}$ can be regarded as a closed set of positive roots.

PROPOSITION 6.1. – *Let $M \simeq \boldsymbol{G}/\boldsymbol{B} \simeq \boldsymbol{U}_0/\boldsymbol{T}_0$ be a complete flag. We keep the notation introduced above.*

(1) *The $\boldsymbol{U}_0$-homogeneous CR structures on M, modulo CR isomorphisms, are in one-to-one correspondence with the set of solvable complex Lie subalgebras $\mathfrak{q}$ of $\mathfrak{g}$ satisfying* (6.2), *modulo automorphisms of $\mathfrak{g}$ which preserve $\mathfrak{b}$.*

(2) *The maximally complex CR structure of M is its standard complex structure, corresponding to the choice $\mathfrak{q} = \mathfrak{b}$, while $\mathfrak{q} = \mathfrak{h}$ yields a totally real M.*

We conclude this subsection by considering *CR* structures that are related to parabolic subalgebras of $\mathfrak{g}$. Recall that a nilpotent subalgebra is *horocyclic* if it is the nilradical of a parabolic subalgebra (cf. [34]).

PROPOSITION 6.2. – *Consider on M the CR structure defined by a CR algebra $(\mathfrak{u}_0, \mathfrak{q})$, with $\mathfrak{q}$ satisfying* (6.2). *Assume that the nilpotent Lie algebra $\mathfrak{n}$ in* (6.1) *is horocyclic and let $\mathfrak{q}'$ be the normalizer of $\mathfrak{n}$ in $\mathfrak{g}$. Then $\mathfrak{q}'$ is parabolic and $\mathfrak{q}' \cap \bar{\mathfrak{q}}'$ is a reductive complement of $\mathfrak{n}$ in $\mathfrak{q}'$:*

$$\mathfrak{q}' = \mathfrak{k} \oplus \mathfrak{n}, \quad \textit{with} \quad \mathfrak{k} = \mathfrak{q}' \cap \bar{\mathfrak{q}}' \ \textit{reductive}, \quad \mathfrak{n} \ \textit{nilpotent}. \tag{6.3}$$

The real Lie algebra $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{u}_0$ is reductive, and $\mathfrak{k}$ is its complexification.

(1) *$(\mathfrak{u}_0, \mathfrak{q}')$ is the CR algebra of a complex flag manifold N.*

(2) *There is a natural $\boldsymbol{U}_0$-equivariant CR fibration $M \xrightarrow{\pi} N$, with totally real fibers. For every $p \in M$, the restriction of $d\pi_p$ defines a $\mathbb{C}$-isomorphism of H_pM with $T_{\pi(p)}N$.*

(3) *M is $\boldsymbol{U}_0$-homogeneous CR-symmetric if and only if N is Hermitian symmetric.*

6.2 – *Symmetric CR structures on complete flags.*

The natural complex structure of the full flag M is not, in general, Hermitian symmetric. We will seek for conditions on the set $\mathcal{Q}$ in (6.1) for which M is $\boldsymbol{U}_0$-*CR*-symmetric. We have

LEMMA 6.3. – *Let* $\mathfrak{q}$ *be defined by* (6.1)*, and assume that* $(\mathfrak{g}_0, \mathfrak{q})$ *defines on* $M = \boldsymbol{U}_0/\boldsymbol{T}_0$ *a* $\boldsymbol{U}_0$*-homogeneous CR-symmetric structure. Then:*

(1) *there exists an involution* λ *of* $\mathfrak{g}$ *with*

$$\lambda(\mathfrak{u}_0) = \mathfrak{u}_0, \qquad \lambda|_{\mathfrak{h}} = \mathrm{Id}, \qquad \lambda|_{\mathfrak{n}} = -\mathrm{Id}; \tag{6.4}$$

(2) $\mathfrak{n}$ *is Abelian and* $\mathfrak{n} + \bar{\mathfrak{n}}$ *generates* $\mathfrak{g}$*, or, equivalently,* $\mathcal{Q}$ *satisfies:*

$$\alpha \in \mathcal{Q} \Longrightarrow -\alpha \notin \mathcal{Q}, \quad \alpha, \beta \in \mathcal{Q} \Longrightarrow \alpha + \beta \notin \mathcal{R}, \tag{6.5}$$

$$\mathcal{R} \subset \mathbb{Z}[\mathcal{Q}]. \tag{6.6}$$

PROOF. – By the assumption, there is an involution λ of $\mathfrak{u}_0$ satisfying (1.16). In particular, λ transforms $\mathfrak{t}_0$ into itself and equals minus the identity on $\big((\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{u}_0\big)/\mathfrak{t}_0$. Its complexification, that we still denote by λ, is an involution of $\mathfrak{g}$ leaving $\mathfrak{h}$ and $\mathfrak{n}$ invariant, hence equal to minus the identity on $\mathfrak{n}$. Since $-[Z_1, Z_2] = \lambda([Z_1, Z_2]) = [\lambda(Z_1), \lambda(Z_2)] = [-Z_1, -Z_2] = [Z_1, Z_2]$ for $Z_1, Z_2 \in \mathfrak{n}$, we get $[\mathfrak{n}, \mathfrak{n}] = \{0\}$, which is equivalent to (6.5).

The conditions that $\mathfrak{q} + \bar{\mathfrak{q}}$ generates $\mathfrak{g}$, that $(\mathfrak{u}_0, \mathfrak{q})$ is fundamental, and that M is of finite type are all equivalent (see § 1.2).

If $\mathfrak{n} + \bar{\mathfrak{n}}$ generates $\mathfrak{g}$, then so does $\mathfrak{q} + \bar{\mathfrak{q}}$. Vice versa, assume that $\mathfrak{q} + \bar{\mathfrak{q}}$ generates $\mathfrak{g}$, and let $\mathfrak{a}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n} + \bar{\mathfrak{n}}$. From $[\mathfrak{h}, \bar{\mathfrak{n}}] = \bar{\mathfrak{n}}$, we obtain that $[\mathfrak{h}, \mathfrak{a}] = \mathfrak{a}$. Hence $\mathfrak{a} + \mathfrak{h} = \mathfrak{g}$. Containing all root spaces, $\mathfrak{a}$ contains also $\mathfrak{h}$ and thus equals $\mathfrak{g}$.

Condition (6.6) is obviously necessary for $(\mathfrak{u}_0, \mathfrak{q})$ to be fundamental. It is also sufficient. Indeed, if $\beta = \sum_{i=1}^{\ell} \varepsilon_i \alpha_i$, with $\alpha_1, \ldots, \alpha_\ell \in \mathcal{Q}$ and $\varepsilon_i = \pm 1$, then, upon reordering, we can assume that $\sum_{i=1}^{h} \varepsilon_i \alpha_i$ is a root for all $1 \leq h \leq \ell$.

Leaving $\mathfrak{h}$ invariant, λ determines an involution λ^* on $\mathcal{R}$, which is the identity on $\mathcal{Q}$. Condition (6.6) implies that $\mathcal{Q}$ spans $\mathfrak{h}^*$, hence λ^* is the identity on $\mathcal{R}$, and therefore $\lambda|_{\mathfrak{h}} = \mathrm{Id}$. □

REMARK 6.4. – When all roots in $\mathcal{R}$ have the same length, orthogonal roots are strongly orthogonal and (6.5) is equivalent to

$$(\alpha|\beta) \geq 0, \quad \forall \alpha, \beta \in \mathcal{Q}. \tag{6.7}$$

According to [14], a $\boldsymbol{U}_0$-CR-symmetric structure on M is *extrinsic* symmetric if there is an isometric embedding of M into a Euclidean space V, and, for every $x \in M$, an isometry of V that restricts to a symmetry of M at x and to the identity on the normal bundle of M at x.

The opposite of the Killing form defines a scalar product on $\mathfrak{u}_0$, which is invariant for the adjoint action of $\boldsymbol{U}_0$. The stabilizer of a regular element X of $\mathfrak{t}_0$ in

$\boldsymbol{U}_0$ is the Cartan subalgebra $\boldsymbol{T}_0$, so that the orbit $\mathrm{Ad}(\boldsymbol{U}_0)(X)$ is an embedding of M. The induced metric is $\boldsymbol{U}_0$-invariant.

The tangent space of M at X is identified, via the differential of the action at the identity, to $\mathfrak{u}_0/\mathfrak{t}_0 \simeq \sum_a \mathfrak{u}_0 \cap (\mathfrak{g}^a + \mathfrak{g}^{-a})$. Under this identification, the subspace $\mathfrak{u}_0 \cap (\mathfrak{g}^a + \mathfrak{g}^{-a})$ is mapped onto itself, and $\mathfrak{t}_0$ is its orthogonal complement in $\mathfrak{u}_0$.

The involution λ of Lemma 6.3 is then an extrinsic symmetry at x. We have proved:

PROPOSITION 6.5. – *If* $M = \boldsymbol{U}_0/\boldsymbol{T}_0$, *endowed with the CR structure defined by the CR algebra* $(\mathfrak{u}_0, \mathfrak{q})$, *where* $\mathfrak{q}$ *is given by* (6.1), *is of finite type and* $\boldsymbol{U}_0$*-CR-symmetric, then it is extrinsic CR-symmetric.* □

6.3 – *CR symmetries, J-properties, and gradings.*

By Lemma 6.3 the involution λ in (6.4) is *inner*. In fact, (6.4) implies that $\lambda = \mathrm{Ad}(\exp(i\pi E))$ for an element

$$E \in \mathcal{R}^\star = \{H \in \mathfrak{h} \mid a(H) \in \mathbb{Z},\ \forall a \in \mathcal{R}\}$$

such that

$$a(E) \equiv 1 \mod 2, \quad \forall a \in \mathcal{Q}. \tag{6.8}$$

The weak-J-property for $(\mathfrak{u}_0, \mathfrak{q})$ will then be equivalent to the possibility of choosing this $E \in \mathcal{R}^\star$ in such a way that

$$a(E) \equiv 1 \mod 4, \quad \forall a \in \mathcal{Q}. \tag{6.9}$$

Indeed, the element J in Definition 1.8 will be equal to $iE \in \mathfrak{t}_0$.

To discuss the symmetric *CR* structures on complete flags in terms of the sets of roots $\mathcal{Q}$, it is convenient to introduce some notation.

DEFINITION 6.6. – *If* $\mathcal{S} \subset \mathcal{R}$, *we shall indicate by* $\mathfrak{Q}(\mathcal{S})$ *the set of all* $\mathcal{Q} \subset \mathcal{S}$ *which satisfy* (6.5) *and* (6.6). *We set* $\mathfrak{Q}_s(\mathcal{S})$ *(resp.* $\mathfrak{Q}_0(\mathcal{S})$, $\mathfrak{Q}_\Upsilon(\mathcal{S})$*) for the subset of* $\mathfrak{Q}(\mathcal{S})$ *consisting of those* $\mathcal{Q} \subset \mathcal{S}$ *for which the* $(\mathfrak{u}_0, \mathfrak{q})$ *with* $\mathfrak{q}$ *given by* (6.1) *is CR-symmetric (resp. has the J-property, has the weak-J-property). We will also say that* $\mathcal{Q}$ *itself is CR-symmetric, or has the J or the weak-J-property when it holds for* $(\mathfrak{u}_0, \mathfrak{q})$. *Clearly* $\mathfrak{Q}_0(\mathcal{S}) \subset \mathfrak{Q}_\Upsilon(\mathcal{S}) \subset \mathfrak{Q}_s(\mathcal{S}) \subset \mathfrak{Q}(\mathcal{S})$.

We indicate by $\mathfrak{Q}'(\mathcal{S})$ *the sets* $\mathcal{Q} \subset \mathcal{S}$ *which satisfy* (6.5).

REMARK 6.7. – For $E \in \mathcal{R}^\star$, $\mathfrak{Q}(\{a \mid a(E) \equiv 1 \bmod 2\}) \subset \mathfrak{Q}_s(\mathcal{R})$ and $\mathfrak{Q}(\{a \mid a(E) \equiv 1 \bmod 4\}) \subset \mathfrak{Q}_\Upsilon(\mathcal{R})$.

Given $\mathcal{Q} \subset \mathcal{R}$, let us define

$$\mathcal{Q}^*_{1,1} = \{\pm(\beta_1 - \beta_2) \in \mathcal{R} \mid \beta_1, \beta_2 \in \mathcal{Q}\}, \tag{6.10}$$

$$\mathcal{Q}^*_h = \Big\{ \sum k_i \beta_i \in \mathcal{R} \Big| \beta_i \in \mathcal{Q},\ k_i \in \mathbb{Z},\ \sum k_i = h \Big\}, \quad h \in \mathbb{Z}. \tag{6.11}$$

We have $\mathcal{Q} \subset \mathcal{Q}^*_1$, $\mathcal{Q}^*_{1,1} \subset \mathcal{Q}^*_0$, and $\mathcal{Q}^*_{-h} = -\mathcal{Q}^*_h$ for all $h \in \mathbb{Z}$. Moreover, $\mathcal{Q}^*_0$ is the root system of the reductive complex Lie subalgebra of $\mathfrak{g}$

$$\mathfrak{q}^{(0)} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{Q}^*_0} \mathfrak{g}^\alpha, \qquad \text{and, with} \tag{6.12}$$

$$\mathfrak{q}^{(h)} = \sum_{\alpha \in \mathcal{Q}^*_h} \mathfrak{g}^\alpha \quad \text{for } h \in \mathbb{Z} \setminus \{0\}, \qquad \text{we have} \tag{6.13}$$

$$[\mathfrak{q}^{(h)}, \mathfrak{q}^{(k)}] \subset \mathfrak{q}^{(h+k)}. \tag{6.14}$$

We have $\mathfrak{g} = \sum_{h \in \mathbb{Z}} \mathfrak{q}^{(h)}$ if, and only if, $\mathcal{R} \subset \mathbb{Z}[\mathcal{Q}]$.

LEMMA 6.8. – *If $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$ and $\mathcal{Q} \cup (-\mathcal{Q}) \neq \mathcal{R}$, then $\mathcal{Q}^*_{1,1} \neq \emptyset$. Moreover,*

$$h_0 \in \mathbb{Z} \setminus \{0\} \text{ and } \mathcal{Q}^*_{1,1} \cap \mathcal{Q}^*_{h_0} = \emptyset \Longrightarrow \mathcal{Q}^*_h \cap \mathcal{Q}^*_{h+h_0} = \emptyset, \forall h \in \mathbb{Z}.$$

PROOF. – Assume that $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$ and that $\mathcal{Q} \cup (-\mathcal{Q}) \neq \mathcal{R}$. Pick $\alpha \in \mathcal{R}$ with $\pm\alpha \notin \mathcal{Q}$. By (6.6), we get $\alpha = \sum_{i=1}^{\ell} \varepsilon_i \beta_i$, with $\ell \geq 2$ and $\varepsilon_i = \pm 1$, $\beta_i \in \mathcal{Q}$ for all $1 \leq i \leq \ell$, and $\sum_{i \leq h} \varepsilon_i \beta_i \in \mathcal{R}$ for all $1 \leq h \leq \ell$. By (6.5) we have $\varepsilon_1 + \varepsilon_2 = 0$, and hence $\varepsilon_1\beta_1 + \varepsilon_2\beta_2 \in \mathcal{Q}^*_{1,1} \neq \emptyset$.

Assume that, for a pair of integers $h \neq k$, the intersection $\mathcal{Q}^*_h \cap \mathcal{Q}^*_k$ contains a root α. Then we can find roots $\beta_i, \gamma_j \in \mathcal{Q}$, and numbers $\varepsilon_i, \eta_j = \pm 1$, for $1 \leq i \leq \ell_1$, $1 \leq j \leq \ell_2$, with $\sum \varepsilon_i = h$, $\sum \eta_j = k$, such that

$$\alpha = \sum_{i=1}^{\ell_1} \varepsilon_i \beta_i = \sum_{j=1}^{\ell_2} \eta_j \gamma_j, \quad \text{with} \quad \begin{cases} \sum_{i=1}^{k} \varepsilon_i \beta_i \in \mathcal{R} & \text{for} \quad k < \ell_1, \\ \sum_{j=1}^{k} \eta_j \gamma_j \in \mathcal{R} & \text{for} \quad k < \ell_2. \end{cases}$$

We can assume that $h \neq \pm 1$. Then $\ell_1 \geq 2$, $\varepsilon_1\beta_1 + \varepsilon_2\beta_2 \in \mathcal{Q}^*_{1,1}$ and, with $h_0 = h - k$, we obtain

$$\varepsilon_1\beta_1 + \varepsilon_2\beta_2 = \sum_{j=1}^{\ell_2} \eta_j \gamma_j - \sum_{i=3}^{\ell_1} \varepsilon_i \beta_i \in \mathcal{Q}^*_{1,1} \cap \mathcal{Q}^*_{h_0}. \qquad \square$$

From (6.6), we obtain

$$\mathfrak{q} = \sum_{h \in \mathbb{Z}} \mathfrak{g}^{(h)}, \quad \text{for } \mathcal{Q} \in \mathfrak{Q}(\mathcal{R}). \tag{6.15}$$

Thus Lemma 6.8 and Remark 6.7 yield criteria for $\mathcal{Q}$ either to be symmetric or to have the J or weak-J-property.

THEOREM 6.9. – *Let* $M = \boldsymbol{U}_0/\boldsymbol{T}_0$ *be a complete flag, with CR structure defined by* $(\mathfrak{u}_0, \mathfrak{q})$*, for a* $\mathfrak{q}$ *given by* (6.1)*, with* $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$*. Then*

(1) *M is $\boldsymbol{U}_0$-CR-symmetric if, and only if,*

$$\mathcal{Q}^*_{1,1} \cap \mathcal{Q}^*_{2h+1} = \emptyset, \quad \textit{for all } h \in \mathbb{Z}. \tag{6.16}$$

(2) *The CR algebra* $(\mathfrak{u}_0, \mathfrak{q})$ *has the weak-J-property if and only if*

$$\mathcal{Q}^*_{1,1} \cap \mathcal{Q}^*_{2h+1} = \emptyset \quad \textit{and} \quad \mathcal{Q}^*_{1,1} \cap \mathcal{Q}^*_{4h+2} = \emptyset, \quad \textit{for all } h \in \mathbb{Z}. \tag{6.17}$$

(3) *The CR algebra* $(\mathfrak{u}_0, \mathfrak{q})$ *has the J-property if and only if* (6.15) *is a* $\mathbb{Z}$*-gradation of* $\mathfrak{g}$.

PROOF. – By Lemma 6.8, conditions (6.16) (resp. (6.17)) implies that $\mathfrak{g}$ admits a $\mathbb{Z}_2$-gradation (resp. a $\mathbb{Z}_4$-gradation) with $\mathfrak{h} \subset \mathfrak{g}_{[0]}$ and $\mathfrak{n} \subset \mathfrak{g}_{[1]}$, where $[a]$ means the congruence class of $a \in \mathbb{Z}$ modulo 2 (resp. modulo 4). Since this gradation is inner, we obtain (1) (resp. (2)).

Finally, if $(\mathfrak{u}_0, \mathfrak{q})$ has the J-property, and (1.15) is valid, then $E = iJ \in \mathcal{R}^\star$ and $[E, Z] = Z$ for all $Z \in \mathfrak{n}$. By (6.6) we get $\mathfrak{g}^{(h)} = \{Z \in \mathfrak{g} \mid [E, Z] = hZ\}$, and (6.15) is a direct sum decomposition, yielding a $\mathbb{Z}$-gradation of $\mathfrak{g}$. Vice versa, if $E \in \mathcal{R}^\star$ defines a $\mathbb{Z}$-gradation with $\alpha(E) = 1$ for all $\alpha \in \mathcal{Q}$, we can take $J = -iE$ to obtain (1.15). □

6.4 – *Complete flags of the classical groups.*

In this section we classify the symmetric *CR* structures on the complete flags of the classical groups.

To fix notation, in the following we shall consider root systems $\mathcal{R} \subset \mathbb{R}^n$, of the types A_{n-1}, B_n, C_n, D_n explicitly described, according to [8], respectively, by:

$$\begin{array}{ll}
(\mathrm{A}_{n-1}) & \mathcal{R} = \{\pm(e_i - e_j) \mid 1 \le i < j \le n\} \subset \mathbb{R}^n, \\
(\mathrm{B}_n) & \mathcal{R} = \{\pm e_i \mid 1 \le i \le n\} \cup \{\pm(e_i \pm e_j) \mid 1 \le i < j \le n\}, \\
(\mathrm{C}_n) & \mathcal{R} = \{\pm 2e_i \mid 1 \le i \le n\} \cup \{\pm(e_i \pm e_j) \mid 1 \le i < j \le n\}, \\
(\mathrm{D}_n) & \mathcal{R} = \{\pm(e_i \pm e_j) \mid 1 \le i < j \le n\}.
\end{array}$$

6.4.1 – Maximal $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$.

We have:

PROPOSITION 6.10. – *Let $\mathcal{R}$ be an irreducible root system of one of the types $\mathrm{A_{n-1}}, \mathrm{B_n}, \mathrm{C_n}, \mathrm{D_n}$. Then, modulo equivalence by the Weyl group $\boldsymbol{W}$ of $\mathcal{R}$, the maximal $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$ are equivalent to one of the following:*

$$(\mathrm{A_{n-1}}) \quad \mathcal{Q}_p = \{e_i - e_j \mid {\scriptstyle 1\le i\le p<j\le n}\}, \quad p = 1, \dots, n-1,$$

$$(\mathrm{B_n}) \quad \begin{cases} \mathcal{Q}_{i_0,p,q_1,\dots,q_s} = \{e_{i_0}\} \cup \{e_i+e_j \mid {\scriptstyle 1\le i<j\le p}\} \\ \qquad\qquad \cup \bigcup_{i=1}^{s} \{e_i \pm e_j \mid {\scriptstyle q_{i-1}<j\le q_i}\} \\ {\scriptstyle 1\le p\le n, \quad 1\le s\le p, \quad q_0=p<q_1<\cdots<q_s=n, \quad q_i+2q_{i-2}\le q_{i-1}, \textit{ for } 2\le i\le s,} \\ {\scriptstyle p=2\Rightarrow s=2, \quad q_{i_0}+q_{i_0-2}<q_{i_0-1} \textit{ if } i_0\ge 2.} \end{cases}$$

$$(\mathrm{C_n}) \qquad \mathcal{Q}_0 = \{2e_i \mid {\scriptstyle 1\le i\le n}\} \cup \{e_i+e_j \mid {\scriptstyle 1\le i<j\le n}\},$$

$$(\mathrm{D_n}) \quad \begin{cases} \mathcal{Q}_{p,q_1,\dots,q_s} = \{e_i+e_j \mid {\scriptstyle 1\le i<j\le p}\} \cup \bigcup_{i=1}^{s} \{e_i \pm e_j \mid {\scriptstyle q_{i-1}<j\le q_i}\}, \\ {\scriptstyle 1\le p\le n,\, 1\le s\le p,\, q_0=p<q_1<\cdots<q_s=n,\, q_i+2q_{i-2}\le q_{i-1}, \textit{ for } 2\le i\le s,\, p=2\Rightarrow s=2,} \\ \mathcal{Q}_{-n} = \{e_i + e_j \mid {\scriptstyle 1\le i<j\le n-1}\} \cup \{e_i - e_n \mid {\scriptstyle 1\le i\le n-1}\}. \end{cases}$$

PROOF. – $(\mathrm{A_{n-1}})$. For $\mathcal{R}$ of type $\mathrm{A_{n-1}}$ and $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$, the sets

$$I = \{i \mid \exists j \text{ s.t. } e_i - e_j \in \mathcal{Q}\} \quad \text{and} \quad I' = \{j \mid \exists i \text{ s.t. } e_i - e_j \in \mathcal{Q}\}$$

are disjoint by (6.5), and by (6.6) they form a partition of $\{1, 2, \dots, n\}$. Then $\mathcal{Q} \subset \{e_i - e_j \mid i \in I,\ j \in I'\}$, and a permutation of $\{1, 2, \dots, n\}$, which corresponds to an element of the Weyl group $\boldsymbol{W}$, transforms $\mathcal{Q}$ into $\mathcal{Q}_p$, for some p with $1 \le p \le n - 1$.

We consider now the case where $\mathcal{R}$ is of one of the types $\mathrm{B_n}$, $\mathrm{C_n}$, $\mathrm{D_n}$. Each $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$ is equivalent, modulo the group $\boldsymbol{A}$ of the isometries of $\mathcal{R}$, to a new $\mathcal{Q}$ with

$$\sup_{\beta\in\mathcal{Q}} (\beta|e_i) > 0 \qquad \text{for} \quad i = 1, \dots, n. \tag{6.18}$$

Indeed $\boldsymbol{A}$ contains all symmetries s_{e_i} for $i = 1, \dots, n$.

Let us single out first the case of $\mathrm{C_n}$. Since $\mathcal{Q}$ cannot contain both $e_i + e_j$ and $e_h - e_j$, condition (6.18) implies that $\mathcal{Q}$ is contained in $\mathcal{Q}_0$.

Let us turn now to $\mathrm{B_n}$ and $\mathrm{D_n}$, and assume that $\mathcal{Q}$ is maximal in $\mathfrak{Q}(\mathcal{R})$. Using conjugation by the Weyl group $\boldsymbol{W}$ to reorder the indices $1, \dots, n$, we can assume that for some integer $p \geq 1$ we have

$$\inf_{\beta\in\mathcal{Q}} (\beta|e_i) \geq 0 \text{ if } 1 \leq i \leq p, \quad \inf_{\beta\in\mathcal{Q}} (\beta|e_i) < 0 \text{ if } p < i \leq n. \tag{6.19}$$

Condition (6.18) implies that $e_i + e_j \in \mathcal{Q}$ for all $1 \leq i < j \leq p$. By (6.5), if $e_i + e_j, e_h - e_j \in \mathcal{Q}$, then $i = h$. Hence, for every $j > p$, there is a unique $i = \lambda(j) \leq p$ such that $e_i - e_j \in \mathcal{Q}$. By reordering, we can assume that $\lambda(\{p+1, \ldots, n\}) = \{1, \ldots, s\}$, that λ is nondecreasing, and that $\#\lambda^{-1}(i) \geq \#\lambda^{-1}(i+1)$ for $1 \leq i < s \leq p$. By maximality, $s = 2$ for $p = 2$. This yields the lists above for $(\mathrm{B_n}), (\mathrm{C_n}), (\mathrm{D_n})$, where we needed to add $\mathcal{Q}_{-n}$ to the list of non equivalent maximal elements for type $\mathrm{D_n}$, because the group $\boldsymbol{A}$ equals $\boldsymbol{W}$ for $\mathrm{B_n}$ and $\mathrm{C_n}$, but contains $\boldsymbol{W}$ as a proper normal subgroup for $\mathrm{D_n}$. □

6.4.2 – Maximal $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{R})$.

Using the results of Proposition 6.10 we characterize, modulo equivalence, all maximal $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{R})$, for $\mathcal{R}$ irreducible of one of the types A, B, C, D.

THEOREM 6.11. – *If $\mathcal{R}$ is an irreducible root system of one of the classical types* $\mathrm{A_{n-1}}$, $\mathrm{B_n}$, $\mathrm{C_n}$, $\mathrm{D_n}$, *then $\mathfrak{Q}_s(\mathcal{R}) = \mathfrak{Q}_0(\mathcal{R})$, i.e. all CR-symmetric* $(\mathfrak{u}_0, \mathfrak{q})$ *have the J-property. Modulo equivalence w.r.t. the Weyl group* $\boldsymbol{W}$ *of $\mathcal{R}$, the maximal $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{R})$ are classified by:*

(A_{n-1}) $\mathfrak{Q}_s(\mathcal{R}) = \mathfrak{Q}(\mathcal{R})$ *and all* $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$ *are maximal.*

(B_n) *Each maximal $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{R})$ is equivalent, modulo* $\boldsymbol{W}$, *to one of the following sets:*

$$\mathcal{Q}'_{i_0,p,q_1,\ldots,q_s} = \{e_{i_0}\} \cup \{e_i + e_j \mid {\scriptstyle 1\leq i\leq s,\ s+1\leq j\leq p}\} \cup \bigcup_{i=1}^{s} \{e_i \pm e_j \mid {\scriptstyle q_{i-1} < j \leq q_i}\},$$

$$1\leq p\leq n, \quad 1\leq s\leq p, \quad q_0=p<q_1<\cdots<q_s=n, \quad q_i+2q_{i-2}\leq q_{i-1}, \text{ for } 2\leq i\leq s,$$
$$p=2\Rightarrow s=2, \quad q_{i_0}+q_{i_0-2}<q_{i_0-1} \text{ if } i_0\geq 2.$$

(C_n) $\mathfrak{Q}_s(\mathcal{R}) = \mathfrak{Q}(\mathcal{R})$.

(D_n) *Any maximal $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{R})$ is isomorphic, modulo* $\boldsymbol{W}$, *to one of the following sets:*

$$\mathcal{Q}_n = \{e_i + e_j \mid {\scriptstyle 1\leq i<j\leq n}\},$$
$$\mathcal{Q}_{-n} = \{e_i + e_j \mid {\scriptstyle 1\leq i<j<n}\} \cup \{e_i - e_n \mid {\scriptstyle 1\leq i<n}\},$$
$$\mathcal{Q}'_{p,q_1,\ldots,q_s} = \{e_i + e_j \mid {\scriptstyle 1\leq i\leq s<j\leq p}\} \cup \bigcup_{i=1}^{s} \{e_i \pm e_j \mid {\scriptstyle q_{i-1}<j\leq q_i}\},$$

$$1\leq p\leq n, \quad 1\leq s\leq p, \quad q_0=p<q_1<\cdots<q_s=n, \quad q_i+2q_{i-2}\leq q_{i-1}, \text{ for } 2\leq i\leq s, \quad p=2\Rightarrow s=2.$$

PROOF. – (A) With the $\mathcal{Q}_p$ defined above, define $J \in \mathfrak{t}_0$ by setting $e_i(J) = i\dfrac{n-p}{n}$ for $1 \leq i \leq p$ and $e_i(J) = -\,i\,\dfrac{p}{n}$ for $p < i \leq n$.

(B) Let $E \in \mathfrak{h}$ define a $\mathbb{Z}_2$-gradation of $\mathfrak{g}$, yielding a CR-symmetry of $(\mathfrak{u}_0, \mathfrak{q})$. Assume that $\mathcal{Q} \subset \mathcal{Q}_{i_0,p,q_1,\dots,q_s}$. Since $e_{i_0}(E) \equiv 1 \bmod 2$, then $e_j(E) \equiv 0 \bmod 2$ for all j for which either $e_{i_0} + e_j \in \mathcal{Q}$, or $e_{i_0} - e_j \in \mathcal{Q}$. In particular, when $s = 0$, $\mathcal{Q}'_{1,n} = \{e_1\} \cup \{e_1 + e_i \mid {\scriptstyle 2 \leq i \leq n}\}$ is contained in $\mathcal{Q}'_{1,1,n} = \{e_1\} \cup \{e_1 \pm e_i \mid {\scriptstyle 2 \leq i \leq n}\}$ and hence is not maximal. Assume therefore that $s \geq 1$. If $1 \leq i < j \leq n$ and $e_i \pm e_j \in \mathcal{Q}$, then $e_i(E) \equiv 1$, $e_j(E) \equiv 0 \bmod 2$. Thus $e_i + e_j \notin \mathcal{Q}$ for $1 \leq i < j \leq s$. Hence we obtain that a maximal $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{R})$ is equivalent to one of the sets listed above. We define the element $J \in \mathfrak{t}_0$ by setting $e_h(J) = i$ for $1 \leq h \leq s$, and $e_h(J) = 0$ for $s < j \leq n$.

(C) With $\mathcal{Q}_0$ defined in Proposition 6.10, we define $e_h(J) = i/2$ for $1 \leq h \leq n$. Then the corresponding $(\mathfrak{u}_0, \mathfrak{q})$ has the J-property.

(D) We can repeat the argument of (B), to conclude that all maximal $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{R})$ are described, modulo equivalence, by the list in ($\mathrm{D_n}$) above. For $\mathcal{Q} = \mathcal{Q}_n$, we define $J \in \mathfrak{t}_0$ by $e_h(J) = i/2$ for all $1 \leq h \leq n$. For $\mathcal{Q} = \mathcal{Q}_{-n}$ we set $e_i(J) = i/2$ for $1 \leq i < n$ and $e_n(J) = -i/2$. For $\mathcal{Q} = \mathcal{Q}'_{p,q_1,\dots,q_s}$, we set $e_h(J) = i$ for $1 \leq h \leq s$, and $e_h(J) = 0$ for $s < h \leq n$. In this way we verify that all maximal $Q \in \mathfrak{Q}_s(\mathcal{R})$, hence all $\mathcal{Q}$ in $\mathfrak{Q}_s(\mathcal{R})$, have the J-property. □

COROLLARY 6.12. – *All CR symmetric $\mathcal{Q}$ contained in a root system of one of the types* A, B, C, D *have the J property.* □

6.5 – *Complete flags of the exceptional groups.*

We turn finally to the complete flags of the exceptional groups.

6.5.1 – Type G_2.

The root system is

$$\mathcal{R} = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq 3\} \cup \{\pm(2e_i - e_j - e_k) \mid (i,j,k) \in \boldsymbol{S}_3\}.$$

According to [33, Theorem 3.11] there is, modulo automorphisms, a unique $\mathbb{Z}_2$-grading of $\mathfrak{g}$, with

$$\begin{aligned} \mathcal{S}_1^4 &= \{\pm(e_1 - e_3), \pm(e_2 - e_3)\} \cup \{\pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3)\},\\ \mathcal{R}_1^4 &= \{\pm(e_1 - e_2)\} \cup \{\pm(2e_3 - e_1 - e_2)\}. \end{aligned}$$

The sum of two short roots is always a root, while the sum of two long roots, if it is a root, is long. Hence a $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$ contains exactly one short root. Modulo isomorphisms, we can assume that $(e_1-e_3)\in\mathcal{Q}$. Then $\mathcal{Q}\subset\{e_1-e_3,2e_1-e_2-e_3,\pm(2e_2-e_1-e_3),e_1+e_2-2e_3\}$. The symmetry with respect to $2e_2-e_1-e_3$ leaves e_1-e_3 invariant and interchanges $2e_1-e_2-e_3$ and $e_1+e_2-2e_3$. Moreover, $(2e_1-e_2-e_3)+(2e_2-e_1-e_3)\in\mathcal{R}$ and $(e_1+e_2-2e_3)-(2e_2-e_1-e_3)\in\mathcal{R}$. Hence, modulo isomorphisms, there are two non equivalent maximal $\mathcal{Q}\in\mathfrak{Q}(\mathcal{R})$:

$$\begin{aligned}\mathcal{Q}^4_1 =&\{e_1-e_3,2e_1-e_2-e_3,e_1+e_3-2e_2\},\\ \mathcal{Q}^4_2 =&\{e_1-e_3,2e_1-e_2-e_3,e_1+e_2-2e_3\}.\end{aligned}$$

Thus we obtain

PROPOSITION 6.13. – *Let $\mathcal{R}$ be simple of type* G_2. *Then:*

(1) *Any maximal $\mathcal{Q}\in\mathfrak{Q}(\mathcal{R})$ is isomorphic either to $\mathcal{Q}^4_1$ or to $\mathcal{Q}^4_2$.*
(2) *$\mathcal{Q}^4_1\in\mathfrak{Q}_s(\mathcal{R})$, and $\mathcal{Q}^4_2\notin\mathfrak{Q}_s(\mathcal{R})$.*
(3) *$\mathfrak{Q}_\Upsilon(\mathcal{R})=\mathfrak{Q}_0(\mathcal{R})$ and all $\mathcal{Q}\in\mathfrak{Q}_\Upsilon(\mathcal{R})$ are isomorphic to*

$$\mathcal{Q}^4_0=\{e_1-e_3,2e_1-e_2-e_3\}.$$

6.5.2 – Type F_4.

We split the root system of type F_4 into two parts, by setting

$$\begin{aligned}\mathcal{R}^4_1 &= \{\pm e_i \mid {\scriptstyle 1\le i\le 4}\}\cup\{\pm e_i\pm e_j\mid {\scriptstyle 1\le i<j\le 4}\},\\ \mathcal{S}^4_1 &= \left\{\pm\frac{1}{2}(e_1\pm e_2\pm e_3\pm e_4)\right\},\\ \mathcal{R} &= \mathcal{R}^4_1\cup\mathcal{S}^4_1.\end{aligned}$$

For the roots of $\mathcal{S}^4_1$ we introduce the notation

$$\beta_0=\frac{1}{2}(e_1+e_2+e_3+e_4),\ \beta_i=\beta_0-e_i,\ \beta_{i,j}=\beta_0-e_i-e_j,\ \text{for } i,j=1,2,3,4.$$

The set $\mathcal{R}^4_1$ is a root system of type B_4. By Proposition 6.10, modulo equivalence, there are five maximal sets in $\mathfrak{Q}(\mathcal{R}^4_1)$, namely:

$$\begin{aligned}\mathcal{Q}^{(4)}_{1,4} =&\{e_1\}\cup\{e_i+e_j\mid {\scriptstyle 1\le i<j\le 4}\},\\ \mathcal{Q}^{(4)}_{1,3,4} =&\{e_1\}\cup\{e_i+e_j\mid {\scriptstyle 1\le i<j\le 3}\}\cup\{e_1\pm e_4\},\\ \mathcal{Q}^{(4)}_{2,3,4} =&\{e_2\}\cup\{e_i+e_j\mid {\scriptstyle 1\le i<j\le 3}\}\cup\{e_1\pm e_4\},\\ \mathcal{Q}^{(4)}_{1,2,3,4} =&\{e_1\}\cup\{e_1+e_2\}\cup\{e_1\pm e_3\}\cup\{e_2\pm e_4\},\\ \mathcal{Q}^{(4)}_{1,1,4} =&\{e_1\}\cup\{e_1\pm e_j\mid {\scriptstyle 2\le j\le 4}\}.\end{aligned}$$

Thus we obtain

PROPOSITION 6.14. – *Modulo equivalence, there are five classes of non equivalent maximal elements of* $\mathfrak{Q}(\mathcal{R})$*, corresponding to the terms of the following list:*

$$\begin{aligned}
\mathcal{Q}^4_{1,4} &= \mathcal{Q}^{(4)}_{1,4} \cup \{\beta_0, \beta_4\},\\
\mathcal{Q}^4_{1,3,4} &= \mathcal{Q}^{(4)}_{1,3,4} \cup \{\beta_0, \beta_4\},\\
\mathcal{Q}^4_{2,3,4} &= \mathcal{Q}^{(4)}_{2,3,4} \cup \{\beta_0, \beta_4\},\\
\mathcal{Q}^4_{1,2,3,4} &= \mathcal{Q}^{(4)}_{1,2,3,4} \cup \{\beta_0, \beta_4\},\\
\mathcal{Q}^4_{1,1,4} &= \mathcal{Q}^{(4)}_{1,1,4} \cup \{\beta_0, \beta_4\}.
\end{aligned}$$

PROOF. – For any choice of three distinct roots in $\mathcal{S}^4_1$, two of them sum to a root. Then a $\mathcal{Q} \in \mathfrak{Q}(\mathcal{R})$ contains at most two roots of $\mathcal{S}^4_1$. The sets in the list are obtained by adding a couple of roots of $\mathcal{S}^4_1$ to each maximal set in $\mathfrak{Q}(\mathcal{R}^4_1)$. Thus they are maximal. The fact that they exhaust the list of maximal elements of $\mathfrak{Q}(\mathcal{R})$ modulo equivalence is proved by considering the set of all roots in $\mathcal{S}^4_1$ that may be added to a $\mathcal{Q}^{(4)}_*$ without contradicting (6.5). $\square$

Modulo equivalence, the maximal elements of $\mathfrak{Q}_s(\mathcal{R}^4_1)$ are

$$\begin{aligned}
\mathcal{Q}'_{1,1,4} &= \{e_1\} \cup \{e_1 \pm e_i \mid {\scriptstyle 2 \leq i \leq 4}\},\\
\mathcal{Q}'_{1,2,3,4} &= \{e_1\} \cup \{e_1 \pm e_3\} \cup \{e_2 \pm e_4\}.
\end{aligned}$$

Thus we obtain

PROPOSITION 6.15. – *We have* $\mathfrak{Q}_s(\mathcal{R}) = \mathfrak{Q}_\Upsilon(\mathcal{R}) = \mathfrak{Q}_0(\mathcal{R})$*, and the maximal elements of* $\mathfrak{Q}_s(\mathcal{R})$ *are all equivalent to*

$$\mathcal{Q}^{4'}_{1,2,3,4} = \{e_1, \beta_0, \beta_4, e_1 \pm e_3, e_2 \pm e_4\}.$$

PROOF. – In fact a maximal $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{R})$ must contain two short roots. Hence, if $E \in \mathfrak{h}$ has integral values on $\mathcal{R}$ and defines a $\mathbb{Z}_2$-gradation with $\alpha(E)$ odd for $\alpha \in \mathcal{Q}$, then there are two even and two odd $e_i(E)$'s. This implies that all maximal elements of $\mathfrak{Q}_s(\mathcal{R})$ are equivalent to $\mathcal{Q}^{4'}_{1,2,3,4} = \mathcal{Q}'_{1,2,3,4} \cup \{\beta_0, \beta_4\}$. We observe that, with $e_1(E) = 1$, $e_2(E) = 1$, $e_3(E) = 0$, $e_4(E) = 0$ we obtain that $\alpha(E) = 1$ for all $\alpha \in \mathcal{Q}$. Hence $\mathcal{Q}^{4'}_{1,2,3,4} \in \mathfrak{Q}_0(\mathcal{R})$. $\square$

6.5.3 – Type E_6, E_7, E_8.

We will write $\mathcal{E}_\ell$ for the root system of type E_ℓ, and we will use an explicit description of [8], with $\mathcal{E}_6 \subset \mathcal{E}_7 \subset \mathcal{E}_8 \subset \mathbb{R}^8$.

It is convenient to use the notation $\beta_\varepsilon = \dfrac{1}{2}\sum_{i=1}^{8} \varepsilon_i e_i$, where $e_1, \ldots, e_8$ is the canonical basis of $\mathbb{R}^8$, and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_8)$, with $\varepsilon_i = \pm 1$ and $\prod_{i=1}^{8} \varepsilon_i = 1$. We shall write sometimes $v_7 = e_8 - e_7$, $v_6 = e_8 - e_7 - e_6$.

For the roots of $\mathcal{S}_1^8$, we shall also employ the simplified notation:

$$\beta_0 = \frac{1}{2}\sum_{i=1}^{8} e_i, \quad \beta_{i,j} = \beta_0 - (e_i + e_j), \quad \beta_{i,j,h,k} = \beta_{i,j} - (e_h + e_k),$$
$$\text{for} \quad 1 \leq i, j, h, k \leq 8 \text{ and pairwise distinct.}$$

Then

$$\begin{aligned}
\mathcal{E}_6 &= \{\pm e_i \pm e_j \mid {\scriptstyle 1\leq i<j\leq 5}\} \cup \{\beta_\varepsilon \mid {\scriptstyle \varepsilon_6=\varepsilon_7=-\varepsilon_8}\},\\
\mathcal{E}_7 &= \{\pm e_i \pm e_j \mid {\scriptstyle 1\leq i<j\leq 6}\} \cup \{\pm(e_7 - e_8)\} \cup \{\beta_\varepsilon \mid {\scriptstyle \varepsilon_7=-\varepsilon_8}\},\\
\mathcal{E}_8 &= \{\pm e_i \pm e_j \mid {\scriptstyle 1\leq i<j\leq 8}\} \cup \{\beta_\varepsilon\},
\end{aligned}$$

Utilizing [33, Chap. 3, § 3.7], we list below the inequivalent $\mathbb{Z}_2$-gradings of the simple complex Lie algebras of type E_ℓ.

Set $\Xi = \{(\ell, i) \mid {\scriptstyle \ell=6,7,8,\ i=1,2}\} \cup \{(7,3)\}$.

For $(\ell, i) \in \Xi$ we denote by $\mathcal{R}_i^\ell$ and $\mathcal{S}_i^\ell$, the set of roots $\alpha \in \mathcal{E}_\ell$ with $\mathfrak{g}^\alpha \subset \mathfrak{g}_{(0)}$ and $\mathfrak{g}^\alpha \subset \mathfrak{g}_{(1)}$, respectively, and we label the grading by the type of $\mathcal{R}_i^\ell$. We added in a third line the definition of an element $E = E_{\ell,i} \in \mathfrak{h}$ yielding the corresponding inner $\mathbb{Z}_2$-gradation.

$$(\mathrm{D}_5) \quad \begin{cases}
\mathcal{R}_1^6 = \{\pm e_i \pm e_j \mid {\scriptstyle 1\leq i<j\leq 5}\},\\
\mathcal{S}_1^6 = \{\beta_\varepsilon \mid {\scriptstyle \varepsilon_6=\varepsilon_7=-\varepsilon_8}\},\\
{\scriptstyle E_{6,1}=E \quad \text{with} \quad e_1(E)=\cdots=e_5(E)=0,\ v_6(E)=2,}
\end{cases}$$

$$(\mathrm{A}_5 \times \mathrm{A}_1) \begin{cases}
\mathcal{R}_2^6 = \{\pm\beta_{6,7}\} \cup \{\pm\beta_{i,8},\ \pm(e_i - e_j) \mid {\scriptstyle 1\leq i\leq 5,\ i<j\leq 5}\},\\
\mathcal{S}_2^6 = \cup\{\pm(e_i + e_j), \pm\beta_{i,j,6,7}, \mid {\scriptstyle 1\leq i<j\leq 5}\},\\
{\scriptstyle E_{6,2}=E \quad \text{with} \quad e_1(E)=\cdots=e_5(E)=\frac{1}{2},\ v_6(E)=\frac{3}{2},}
\end{cases}$$

$$(\mathrm{D}_6 \times \mathrm{A}_1) \begin{cases}
\mathcal{R}_1^7 = \{\pm e_i \pm e_j \mid {\scriptstyle 1\leq i<j\leq 6}\} \cup \{\pm v_7\},\\
\mathcal{S}_1^7 = \{\beta_\varepsilon \mid {\scriptstyle \varepsilon_7=-\varepsilon_8}\},\\
{\scriptstyle E_{7,1}=E \quad \text{with} \quad e_1(E)=\cdots=e_6(E)=0,\ v_7(E)=2,}
\end{cases}$$

$$(\mathrm{A}_7) \quad \begin{cases}
\mathcal{R}_2^7 = \{\pm v_7\} \cup \{\pm(e_i - e_j), \pm\beta_{i,7}, \pm\beta_{i,8} \mid {\scriptstyle 1\leq i\leq 6,\ i<j\leq 6}\},\\
\mathcal{S}_2^7 = \{\pm(e_i + e_j), \beta_{i,j,h,7}, \beta_{i,j,h,8} \mid {\scriptstyle 1\leq i<j\leq 6,\ j<h\leq 6}\},\\
{\scriptstyle E_{7,2}=E \quad \text{with} \quad e_1(E)=\cdots=e_6(E)=\frac{1}{2},\ v_7(E)=2,}
\end{cases}$$

$$(\mathrm{E}_6) \quad \begin{cases}
\mathcal{R}_3^7 = \{\beta_{6,7}\} \cup \left\{\pm\beta_{i,8}, \beta_{i,j,h,8}, \beta_{i,j,6,7}, \pm e_i \pm e_j \Big|_{\substack{1\leq i\leq 5,\\ i<j\leq 5,\ j<h\leq 5}}\right\}\\
\mathcal{S}_3^7 = \{\pm v_7, \pm\beta_{6,8}\} \cup \left\{\pm e_i \pm e_6, \pm\beta_{i,7}, \beta_{i,j,h,7}, \beta_{i,j,6,8} \Big|_{\substack{1\leq i\leq 5,\\ i<j\leq 5,\ j<h\leq 5}}\right\}\\
{\scriptstyle E_{7,3}=E \quad \text{with} \quad e_1(E)=\cdots=e_5(E)=0,\ e_6(E)=1,\ v_7(E)=1,}
\end{cases}$$

$$(\mathrm{D}_8)\quad \begin{cases} \mathcal{R}^8_1=\{\pm e_i\pm e_j \mid {\scriptstyle 1\le i<j\le 8}\},\\ \mathcal{S}^8_1=\{\beta_\varepsilon\in\mathcal{R}\},\\ {\scriptstyle E_{8,1}=E \quad \text{with} \quad e_1(E)=\cdots=e_7(E)=0,\ e_8(E)=2,} \end{cases}$$

$$(\mathrm{E}_7\times\mathrm{A}_1)\begin{cases} \mathcal{R}^8_2=\{\pm(e_i-e_j),\beta_{i,j,h,k}\mid {\scriptstyle 1\le i<j\le 8,\, j<h<k\le 8}\}\cup\{\pm\beta_0\}\\ \mathcal{S}^8_2=\{\pm(e_i+e_j)\mid {\scriptstyle 1\le i<j\le 8}\}\cup\{\pm\beta_{i,j}\mid {\scriptstyle 1\le i<j\le 8}\}\\ {\scriptstyle E_{8,2}=E \quad \text{with} \quad e_1(E)=\cdots=e_8(E)=\frac{1}{2}.} \end{cases}$$

We denote by $\boldsymbol{W}^\ell_i$ the Weyl group of $\mathcal{R}^\ell_i$. For $(\ell,i)\in\varXi\setminus\{(7,3)\}$, it coincides with the Weyl group of $\mathcal{R}^\ell_i$. The Weyl group of $\mathcal{R}^7_3$ is a normal subgroup of index two of $\boldsymbol{W}^7_3$.

EXAMPLE 6.16. – Consider the set

$$\mathcal{Q}=\{\beta_0,\beta_{1,2},\beta_{1,3},\beta_{2,3},\beta_{4,5}\}\cup\{\beta_{j,h},\beta_{h,k}\mid {\scriptstyle j=4,5,\ 6\le h\le 8,\ h<k\le 8}\}.$$

One can show that $\mathcal{Q}\in\mathfrak{Q}_\Upsilon(\mathcal{E}_8)\setminus\mathfrak{Q}_0(\mathcal{E}_8)$, but is not maximal in $\mathfrak{Q}_s(\mathcal{E}_8)$.

LEMMA 6.17. – (1) *Let* $\ell\in\{6,7,8\}$. *For every* $\alpha\in\mathcal{Q}\in\mathfrak{Q}(\mathcal{E}_\ell)$ *we can find* $\beta\in\mathcal{Q}$, $\beta\neq\alpha$, *with* $(\alpha|\beta)>0$.

(2) *Let* $(\ell,i)\in\varXi$, *and* $\alpha_0,\alpha_1,\alpha_2\in\mathcal{S}^\ell_i$, *then*

$$(\alpha_0|\alpha_1)>0,\ (\alpha_0|\alpha_2)>0 \implies \alpha_1+\alpha_2\notin\mathcal{R}. \tag{6.20}$$

(3) *For* $(\ell,i)\in\varXi\setminus\{(6,1),(7,3)\}$ *the group* $\boldsymbol{W}^\ell_i$ *is transitive on* $\mathcal{S}^\ell_i$. *The set* $\mathcal{S}^6_1$ *is the union of the two orbits of* $\boldsymbol{W}^6_1$:

$$\mathcal{Q}_{6,1,\beta_{6,7},-\beta_{1,8}}=\{\beta_{6,7}\}\cup\{-\beta_{i,8},\beta_{i,j,6,7}\mid {\scriptstyle 1\le i\le 5,\ i<j\le 5}\},$$
$$\mathcal{Q}_{6,1,-\beta_{6,7},\beta_{1,8}}=\{-\beta_{6,7}\}\cup\{\beta_{i,8},\beta_{i,j,h,8}\mid {\scriptstyle 1\le i\le 5,\ i<j<h\le 5}\},$$

that are transformed one into the other by an outer automorphism of $\mathcal{R}^6_1$. *Both are maximal in* $\mathfrak{Q}(\mathcal{E}_6)$ *and belong to* $\mathfrak{Q}_0(\mathcal{E}_6)$.

The set $\mathcal{S}^7_3$ *is the union of the two* $\boldsymbol{W}^7_3$*-orbits*

$$\mathcal{Q}_{7,3,v_7,e_1+e_6}=\{v_7,-\beta_{6,8}\}\cup\{\pm e_i+e_6,\beta_{i,j,h,7}\mid {\scriptstyle 1\le i\le 5,\ i<j<h\le 5}\}$$
$$\mathcal{Q}_{7,3,-v_7,e_1-e_6}=\{-v_7,\beta_{6,8}\}\cup\{\pm e_i-e_6,\beta_{i,j,6,8}\mid {\scriptstyle 1\le i\le 5,\ i<j\le 5}\}.$$

They are transformed one into the other by an outer automorphism of $\mathcal{R}^7_3$, *that is an element of* $\boldsymbol{W}^7_3$. *They are maximal in* $\mathfrak{Q}(\mathcal{E}_7)$ *and belong to* $\mathfrak{Q}_0(\mathcal{E}_7)$.

PROOF. – Since all roots in $\mathcal{E}_\ell$ have equal length, orthogonal roots are strongly orthogonal. Thus, if $\alpha\in\mathcal{Q}$ is orthogonal to $\mathcal{Q}\setminus\{\alpha\}$, then $\mathcal{E}_\ell\cap\mathbb{Z}[\mathcal{Q}]$ decomposes into $\{\pm\alpha\}$ and $\mathbb{Z}[\mathcal{Q}\setminus\{\alpha\}]$. Hence $\mathcal{E}_\ell\cap\mathbb{Z}[\mathcal{Q}]\neq\mathcal{E}_\ell$, because $\mathcal{E}_\ell$ is irreducible. This proves (1).

It suffices to prove (2) in the case a_0, a_1, a_2 are distinct. Then, by the assumption, $(a_0|a_1) = (a_0|a_2) = 1$. If a_1+a_2 is a root, then $(a_1|a_2) = -1$. Hence $(a_0 - a_1|a_2) = 2$ yields $a_2 = a_0 - a_1$. Therefore there is no $E \in \mathcal{E}_\ell^*$ with $a_i(E)$ odd for $i = 0, 1, 2$.

For (ℓ, i) equal to either $(6,1)$ or $(7,3)$, the element $E = E_{\ell,i} \in \mathfrak{h}$ satisfies $a(E) = 0$ for all $a \in \mathcal{R}_i^\ell$. Hence in this two cases $\mathcal{S}_i^\ell$ splits into $\{a \in \mathcal{S}_i^\ell \mid a(E) = \lambda\}$, for $\lambda = \pm 1$, and each of the two sets belongs to $\mathfrak{Q}_s(\mathcal{E}_\ell)$. For $(\ell, i) \in \Xi \setminus \{(6,1),(7,3)\}$ the transitivity of $\boldsymbol{W}_i^\ell$ on $\mathcal{S}_i^8$ can be easily checked by a case by case verification. $\square$

Proposition 6.18. – *For $(\ell, i) \in \Xi \setminus \{(6,1),(7,3)\}$, and $a_0 \in \mathcal{S}_i^\ell$, the set*

$$\mathcal{Q}_{\ell,i,a_0} = \{a \in \mathcal{S}_i^\ell \mid (a|a_0) > 0\} \tag{6.21}$$

is a maximal element of $\mathfrak{Q}_s(\mathcal{E}_\ell)$ and does not belong to $\mathfrak{Q}_\Upsilon(\mathcal{E}_\ell)$.

Proof. – For each $(\ell, i) \in \Xi \setminus \{(6,1),(7,3)\}$, the Weyl group of $\mathcal{R}_i^\ell$ is transitive on $\mathcal{S}_i^\ell$. Hence it suffices to consider $\mathcal{Q}_{\ell,i,a_0}$ when a_0 is any specific element of $\mathcal{S}_i^\ell$. We have:

$$\begin{aligned}
\mathcal{Q}_{6,2,e_4+e_5} &= \{e_4+e_5, \beta_{1,2,3,8}\} \cup \{e_i + e_r, \beta_{i,j,6,7} \mid {\scriptstyle 1\le i\le r,\, i<j\le 3,\, r=4,5}\},\\
\mathcal{Q}_{7,1,\beta_{6,7}} &= \{\beta_{6,7}, \beta_{6,8}\} \cup \{\beta_{i,7}, \beta_{i,j,6,7} \mid {\scriptstyle 1\le i\le 5,\, i<j\le 5}\},\\
\mathcal{Q}_{7,2,e_5+e_6} &= \{e_5+e_6\} \cup \{e_i+e_k, \beta_{i,j,h,r} \mid {\scriptstyle 1\le i\le 4,\, i<j<h\le 4,\, k=5,6,\, r=7,8}\},\\
\mathcal{Q}_{8,1,\beta_0} &= \{\beta_0\} \cup \{\beta_{i,j} \mid {\scriptstyle 1\le i<j\le 8}\},\\
\mathcal{Q}_{8,2,e_7+e_8} &= \{e_7+e_8, -\beta_{7,8}\} \cup \{e_i+e_r, \beta_{i,j} \mid {\scriptstyle 1\le i\le 6,\, i<j\le 6,\, r=7,8}\}.
\end{aligned}$$

We give the complete proof for the case $(8,1)$. The other cases can be discussed similarly.

First we note that $e_i + e_j = \beta_0 - \beta_{i,j}$, $e_i - e_j = \beta_{j,h} - \beta_{i,h}, \beta_{i,j,h,k} = \beta_{i,j} + \beta_{h,k} - \beta_0 \in \mathbb{Z}[\mathcal{Q}_{8,1,\beta_0}]$ for all four-tuple i, j, h, k of distinct indices with $1 \le i, j, h \le 8$ shows that $\mathcal{E}_8 \subset \mathbb{Z}[\mathcal{Q}_{8,1,\beta_0}]$. Moreover, $(\beta_{i,j}|e_i+e_j) = -1$, $(\beta_{i,h}|e_i-e_j) = -1$, $(\beta_0|-e_i-e_j) = -1$, $(\beta_{i,j}|\beta_{h,k,r,s}) = -1$ for all sets i, j, h, k, r, s of distinct indices with $1 \le i, j, h, k, r, s \le 8$ shows that $\mathcal{Q}_{8,1,\beta_0}$ is maximal in $\mathfrak{Q}(\mathcal{E}_8)$.

Let us show that $\mathcal{Q}_{1,\beta_0}^8 \notin \mathfrak{Q}_\Upsilon(\mathcal{R})$. We argue by contradiction. From $\beta(E) \equiv 1 \bmod 4$ for all $\beta \in \mathcal{Q}_{1,\beta_0}^8$ we obtain that $e_i(E) \pm e_j(E) \equiv 0 \bmod 4$ for $1 \le i < j \le 8$. Then $e_i(E) = 2k_i$ is an even integer for all $i = 1, \dots, 8$. But then $\beta_0(E) \equiv 1 \bmod 4$ and $\beta_{i,j}(E) \equiv 1 \bmod 4$ imply that at the same time $\sum_{i=1}^{8} k_1 \equiv 1$ and $2(k_i + k_j) \equiv 0 \bmod 4$, yielding a contradiction, since the second set of equations tells us that the k_i's are either all odd, or all even. $\square$

Example 6.19. – Consider, for an integer p with $1 \le p \le 8$, the set

$$\mathcal{Q}'_p = \{\beta_0\} \cup \{e_i+e_r, \beta_{i,j}, \beta_{r,s} \mid {\scriptstyle 1\le i\le p,\, i<j\le p,\, p<r\le 8,\, r<s\le 8}\} \subset \mathcal{E}_8.$$

We note that $\mathcal{Q}'_p \simeq \mathcal{Q}'_{8-p}$ for $p = 1, \ldots, 8$. Each $\mathcal{Q} \in \mathfrak{Q}(\mathcal{E}_8)$ which is maximal and is contained in $\{\alpha \in \mathcal{E}_8 \mid (\beta_0|\alpha) > 0\}$ is equivalent, modulo $\boldsymbol{W}$, to some $\mathcal{Q}'_p$. One easily verifies that $\mathcal{Q}'_p \notin \mathfrak{Q}_s(\mathcal{E}_8)$ for p odd and $\mathcal{Q}'_p \in \mathfrak{Q}_s(\mathcal{E}_8)$ for p even.

The definition in (6.21) can be generalized in the following way:

PROPOSITION 6.20. – *Let $(\ell, i) \in \Xi$ be fixed. Define, for*

$$a_1, \ldots, a_k \in \mathcal{S}_i^\ell\,, \text{ with } \inf_{1 \le j < h \le k} (a_j|a_h) \ge 0, \tag{6.22}$$

$$\begin{cases} \mathcal{Q}^0_{\ell,i,a_1,\ldots,a_k} = \{a_1, \ldots, a_k\}, \\ \mathcal{Q}^j_{\ell,i,a_1,\ldots,a_k} = \mathcal{Q}_{\ell,i,a_j} \cap \{\alpha \in \mathcal{S}_i^\ell \mid (\alpha|\beta) \ge 0,\ \forall \beta \in \mathcal{Q}^{j-1}_{\ell,i,a_1,\ldots,a_k}\} \quad {\scriptstyle (1 \le j \le k),} \\ \mathcal{Q}_{\ell,i,a_1,\ldots,a_k} = \mathcal{Q}^k_{\ell,i,a_1,\ldots,a_k}. \end{cases}$$

Then $\mathcal{Q}_{\ell,i,a_1,\ldots,a_k} \in \mathfrak{Q}'(\mathcal{S}_i^\ell)$. If $a_1 \ldots, a_k$ contains a basis of $\mathbb{R}[\mathcal{E}_\ell]$, then $\mathcal{Q}_{\ell,i,a_1,\ldots,a_k}$ is a maximal element of $\mathfrak{Q}'(\mathcal{S}_i^\ell)$.

PROOF. – Indeed, using (6.20) we prove by recurrence on $j = 0, 1, \ldots, k$ that $(\alpha'|\alpha'') \ge 0$ for all $\alpha', \alpha'' \in \mathcal{Q}^j_{\ell,i,a_1,\ldots,a_k}$. Assume now that $\mathcal{E}_\ell \subset \mathbb{R}[\mathcal{Q}_{\ell,i,a_1,\ldots,a_k}]$. If $\alpha \in \mathcal{S}_i^\ell$ satisfies $(\alpha|\beta) \ge 0$ for all $\beta \in \mathcal{Q}_{\ell,i,a_1,\ldots,a_k}$ there is a j, with $1 \le j \le k$, such that $(\alpha|a_j) > 0$. Then $\alpha \in \mathcal{Q}^j_{\ell,i,a_1,\ldots,a_k}$. □

EXAMPLE 6.21. – Let $(\ell, i) \in \Xi \setminus \{(6,1), (7,3)\}$. Then, if $a_1, a_2 \in \mathcal{S}_i^\ell$ and $(a_1|a_2) > 0$, we have $\mathcal{Q}_{a_1,a_2} = \mathcal{Q}_{a_1}$. Thus the maximal $\mathcal{Q} \in \mathfrak{Q}(\mathcal{S}_i^8)$ that can be described by a sequence (6.22) with $1 \le k \le 2$ are equivalent either to $\mathcal{Q}_{\ell,i,a}$ or to $\mathcal{Q}_{a_1,a_2}$ with $a, a_1, a_2 \in \mathcal{S}_i^\ell$ and $(a_1|a_2) = 0$. Since $\boldsymbol{W}_i^\ell$ is transitive on the pair of orthogonal roots, we obtain that each of these sets is equivalent to one of the following:

$$\mathcal{Q}_{6,2,e_1+e_5,e_2+e_4} = \{\beta_{2,3,6,7}, \beta_{3,4,6,7}\} \cup \{e_i + e_j \mid {\scriptstyle 1 \le i < j \le 5,\ (i,j) \ne (2,3),(3,4)}\} \in \mathfrak{Q}_0(\mathcal{E}_6),$$

$$\mathcal{Q}_{7,1,\beta_{6,8},\beta_{1,7},\beta_{1,8}} = \{\beta_{1,7}, \beta_{6,7}\} \cup \{\beta_{i,8}, \beta_{1,i,6,8} \mid {\scriptstyle 2 \le i \le 5}\} \in \mathfrak{Q}_s(\mathcal{E}_7) \setminus \mathfrak{Q}_\Upsilon(\mathcal{E}_7),$$

$$\begin{aligned} \mathcal{Q}_{7,2,e_1+e_2,e_3+e_4} = \{&e_3 + e_4, \beta_{3,5,6,7}, \beta_{4,5,6,7}, \beta_{3,5,6,8}, \beta_{4,5,6,8}\} \\ &\cup \{e_i + e_j \mid {\scriptstyle i=1,2,\ i<j \le 6}\} \in \mathfrak{Q}_s(\mathcal{E}_7) \setminus \mathfrak{Q}_\Upsilon(\mathcal{E}_7), \end{aligned}$$

$$\mathcal{Q}_{8,1,\beta_0,\beta_{1,2,3,4}} = \{\beta_0, \beta_{1,2,3,4}\} \cup \{\beta_{i,j} \mid {\scriptstyle 1 \le i \le 4,\ i<j \le 8}\} \in \mathfrak{Q}_s(\mathcal{E}_8) \setminus \mathfrak{Q}_\Upsilon(\mathcal{E}_8),$$

$$\begin{aligned} \mathcal{Q}_{8,2,\beta_{7,8},\beta_{5,6}} = \{&\beta_{5,6}, \beta_{7,8}\} \cup \{\beta_{i,r} \mid {\scriptstyle 1 \le i \le 6,\ r=7,8}\} \\ &\cup \{e_i + e_j \mid {\scriptstyle 1 \le i < j \le 6,\ (i,j) \ne (5,6)}\} \in \mathfrak{Q}_0(\mathcal{E}_8). \end{aligned}$$

Let a_i, a_2 be two orthogonal roots in $\mathcal{S}_i^\ell$. Then $\mathcal{Q}^\ell_{a_1,a_2} \in \mathfrak{Q}_0(\mathcal{E}_\ell)$ for $(\ell, i) \in \{(6,2), (8,2)\}$, and $\mathcal{Q}^\ell_{a_1,a_2} \in \mathfrak{Q}_s(\mathcal{E}_\ell) \setminus \mathfrak{Q}_\Upsilon(\mathcal{E}_\ell)$ for $(\ell, i) \in \{(7,1), (7,2), (8,1)\}$.

EXAMPLE 6.22. – The set

$$\mathcal{Q}_{8,1,\beta_0,\beta_{1,2,3,4},\beta_{1,2,3,5}} = \{\beta_0, \beta_{4,5}\} \cup \{\beta_{i,r}, \beta_{i,j,h,k} \mid {\scriptstyle 1 \le i \le 3,\ i<r \le 8,\ i<j \le 3,\ j<h<k \le 8}\}$$

cannot be represented as $\mathcal{Q}_{8,1,a_1,\dots,a_k}$ for a sequence of orthogonal roots $a_1,\dots,a_k \in \mathcal{S}_1^8$. It belongs to $\mathfrak{Q}_s(\mathcal{E}_8) \setminus \mathfrak{Q}_\Upsilon(\mathcal{E}_8)$.

EXAMPLE 6.23. – We consider below various examples of maximal $\mathcal{Q} \in \mathfrak{Q}_i^8$, $i = 1, 2$, defined by sequences of three or more orthogonal roots. Example (5) shows that there is a maximal element $\mathcal{Q}$ of $\mathfrak{Q}_s(\mathcal{E}_\ell)$ in $\mathfrak{Q}_\Upsilon(\mathcal{E}_8) \setminus \mathfrak{Q}_0(\mathcal{E}_8)$.

$$\begin{aligned}(1)\qquad \mathcal{Q}_{8,1,\beta_0,\beta_{1,2,3,4},\beta_{1,2,5,6}} &= \{\beta_0, \beta_{1,2}, \beta_{1,2,3,4}, \beta_{1,2,5,6}\} \\ &\cup \left\{\beta_{i,j}, \beta_{i,h}, \beta_{i,k}, \beta_{j,h}, \beta_{1,2,j,h}, \beta_{i,3,4,h}, \beta_{i,j,5,6} \,\middle|\, {}^{i=1,2,\ j=3,4,}_{h=5,6,\ k=7,8}\right\}.\end{aligned}$$

This set belongs to $\mathfrak{Q}_s(\mathcal{E}_8) \setminus \mathfrak{Q}_\Upsilon(\mathcal{E}_8)$.

$$\begin{aligned}(2)\qquad \mathcal{Q} &= \mathcal{Q}_{8,1,\beta_{1,2,3,4},\beta_{1,3,5,6},\beta_{1,3,5,8}} \\ &= \{\beta_0, \beta_{1,2,3,4}, \beta_{1,3,5,6}, \beta_{1,3,5,8}, \beta_{1,2,3,5}\} \cup \{\beta_{1,i} \mid 2 \le i \le 8\} \\ &\cup \{\beta_{2,3}, \beta_{2,5}, \beta_{2,8}, \beta_{3,5}, \beta_{3,6}, \beta_{4,5}\}.\end{aligned}$$

Then $\mathcal{Q} \in \mathfrak{Q}_s(\mathcal{E}_8) \setminus \mathfrak{Q}_\Upsilon(\mathcal{E}_8)$, and is maximal.

$$(3)\qquad \mathcal{Q} = \mathcal{Q}_{8,1,\beta_0,\beta_{1,2,3,4},\beta_{1,2,5,6},\beta_{3,4,5,6},\beta_{1,3,5,7}}$$

We claim that $\mathcal{Q} \in \mathfrak{Q}_0(\mathcal{E}_8)$. Indeed, $(\beta_{r,s}|\beta_{i,j,h,k}) \ge 0$ if and only if $\{r,s\} \cap \{i,j,h,k\} \neq \emptyset$, hence

$$\{1,2,3,4\} \cap \{1,2,5,6\} \cap \{3,4,5,6\} \cap \{1,3,5,7\} = \emptyset \Longrightarrow \beta_{i,8} \notin \mathcal{Q}, \text{ for } i = 1,\dots,7.$$

The conditions $(\beta_{r,s}|\beta_{i,j,h,k}) \ge 0$, $(\beta_{a,b,c,d}|\beta_{i,j,h,k}) \ge 0$ are equivalent to $\{r,s\} \cap \{i,j,h,k\} \neq \emptyset$, $\#\{a,b,c,d\} \cap \{i,j,h,k\} \ge 2$, respectively. Moreover,

$$\{\beta_{1,3}, \beta_{1,4}, \beta_{1,5}, \beta_{1,6}, \beta_{2,3}, \beta_{2,5}, \beta_{3,5}, \beta_{3,6}, \beta_{4,5}, \beta_{4,6}, \beta_{4,7}\} \subset \mathcal{Q}.$$

Then we can easily show that $\mathcal{Q}$ cannot contain any root of type $\beta_{i,j,h,8}$ with $1 \le i < j < h \le 7$.

Therefore $\alpha(E_{8,1}) = 1$ for all $\alpha \in \mathcal{Q}_{\beta_0,\beta_{1,2,3,4},\beta_{1,2,5,6},\beta_{3,4,5,6},\beta_{1,3,5,7}}$.

$$\begin{aligned}(4)\qquad \mathcal{Q} &= \mathcal{Q}_{8,1,\beta_0,\beta_{1,2,3,4},\beta_{1,2,5,6},\beta_{1,3,6,7},\beta_{2,3,6,8},\beta_{2,3,5,7},\beta_{1,3,5,8}} \\ &= \{\beta_0, \beta_{1,2}, \beta_{1,3}, \beta_{2,3}, \beta_{3,5}, \beta_{3,6}, \beta_{1,2,3,4}, \beta_{1,2,3,5}, \beta_{1,2,3,6}, \beta_{1,2,3,7}, \beta_{1,2,3,8}, \\ &\quad \beta_{1,2,5,6}, \beta_{1,3,5,6}, \beta_{1,3,6,7}, \beta_{2,3,5,6}, \beta_{2,3,6,8}, \beta_{1,3,6,8}, \beta_{2,3,5,7}, \beta_{1,3,5,8}\}\end{aligned}$$

is maximal in $\mathfrak{Q}(\mathcal{E}_8)$ and belongs to $\mathfrak{Q}_s(\mathcal{E}_8) \setminus \mathfrak{Q}_\Upsilon(\mathcal{E}_8)$.

$$\begin{aligned}(5)\qquad \mathcal{Q} &= \mathcal{Q}_{8,1,\beta_0,\beta_{1,2,6,7},\beta_{3,4,6,7},\beta_{1,3,6,8},\beta_{2,4,6,8},\beta_{1,4,7,8},\beta_{2,3,7,8}} \\ &= \{\beta_0, \beta_{6,7}, \beta_{6,8}, \beta_{7,8}, \beta_{1,2,6,7}, \beta_{3,4,6,7}, \beta_{1,3,6,8}, \beta_{2,4,6,8}, \beta_{1,4,7,8}, \\ &\quad \beta_{1,6,7,8}, \beta_{2,3,7,8}, \beta_{2,6,7,8}, \beta_{3,6,7,8}, \beta_{4,6,7,8}\}\end{aligned}$$

is a maximal element of $\mathfrak{Q}(\mathcal{S}_1^8)$, and $a(E) \equiv 1 \bmod 4$ for $E \in \mathfrak{h}$ defined by $e_1(E) = \cdots = e_5(E) = 0, e_6(E) = e_7(E) = e_8(E) = -2$. Moreover, $\mathcal{Q} \in \mathfrak{Q}_\Upsilon(\mathcal{R}) \setminus \mathfrak{Q}_0(\mathcal{R})$.

$$\mathcal{Q}_{8,2,\beta_{7,8},\beta_{5,6},\beta_{3,4}} = \{\beta_{3,4}, \beta_{5,6}\} \cup \{\beta_{i,r} {}_{1\le i\le 6,\, r=7,8}\} \cup \{e_i + e_j \mid {}_{1\le i<j\le 6,\, (i,j)\neq(3,4),(5,6)}\} \tag{6}$$

is also maximal and belongs to $\mathfrak{Q}_0(\mathcal{E}_8)$.

$$\mathcal{Q}_{8,2,\beta_{7,8},\beta_{5,6},\beta_{3,4},\beta_{1,2}} = \{\beta_{1,2}, \beta_{3,4}, \beta_{5,6}, \beta_{7,8}\} \cup \{\beta_{i,r} {}_{1\le i\le 6,\, r=7,8}\} \cup \{e_i + e_j \mid {}_{1\le i<j\le 6,\, (i,j)\neq(1,2),(3,4),(5,6)}\} \tag{7}$$

is maximal and belongs to $\mathfrak{Q}_0(\mathcal{E}_8)$.

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