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ON THE TOPOLOGY OF MINIMAL ORBITS IN COMPLEX FLAG MANIFOLDS

ANDREA ALTOMANI, COSTANTINO MEDORI AND MAURO NACINOVICH

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Abstract. We compute the Euler-Poincaré characteristic of the homogeneous compact manifolds that can be described as minimal orbits for the action of a real form in a complex flag manifold.

1. Introduction. A *complex flag manifold* is a simply connected homogeneous compact complex manifold that is also a projective variety. It is the quotient $\hat{M} = \hat{\mathbf{G}}/\mathbf{Q}$ of a connected complex semisimple Lie group $\hat{\mathbf{G}}$ by a parabolic subgroup $\mathbf{Q}$. Let a connected real form $\mathbf{G}$ of $\hat{\mathbf{G}}$ act on $\hat{M}$ by left translations. This action decomposes $\hat{M}$ into a finite number of $\mathbf{G}$-orbits. Among these, there is a unique orbit of minimal dimension, which is also the only one that is compact (cf. [Wol69]).

In this paper we compute the Euler-Poincaré characteristic of the minimal orbit M. This was already well known in the two cases where either $M = \hat{M}$, i.e., when $\mathbf{G}$ is transitive on $\hat{M}$, or M is totally real, i.e., when $\mathbf{Q} \cap \mathbf{G}$ is a real form of $\mathbf{Q}$, and, in particular, a real parabolic subgroup of $\mathbf{G}$. In these cases, indeed, explicit cell decompositions of M were obtained by several authors (see, e.g., [CS99, DKV83, Koc95]). The Euler characteristic of M was also computed in [MN01] for the case where M is a *standard* CR manifold. These are indeed special cases of minimal orbits, in which, although $\mathbf{Q} \cap \mathbf{G}$ is not a real form of $\mathbf{Q}$, M is diffeomorphic to a *real* flag manifold.

Our treatment of the general case, here, utilizes several notions developed in [AMN06a] for the study of the CR geometry of the minimal orbits. As in that paper, we shall use their representation in terms of the cross-marked Satake diagrams associated to their *parabolic* CR *algebras*. This makes easier to deal effectively with their $\mathbf{G}$-equivariant fibrations, by reducing the computation of the structure of the fibers to combinatorics on the Satake diagrams.

After observing that we may reduce to the case where $\mathbf{G}$ is simple, we show that in this case the Euler characteristic is different from zero, and hence positive, when $\mathbf{G}$ is compact, or of the complex type (in these cases M is diffeomorphic to a complex flag manifold), or of the real types A II, D II and E IV and for some special real flag manifolds of the real types A I, D I and E I. We explicitly compute $\chi(M)$ when $\mathbf{Q}$ is maximal parabolic and explain how, to compute $\chi(M)$ for general M, we may always reduce to that special case.

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The paper is organized as follows. In Sections 2 and 3 we rehearse the basic notions on complex flag manifolds and minimal orbits, and prove some results about **G**-equivariant fibrations. In Section 4 we establish some general criteria and tools that will be used to compute the Euler characteristic of the minimal orbits, and then in Section 5 we prove our main results. In Section 6 we further illustrate our method through the discussion of some examples. The final section is an appendix, containing a table that collects all the basic information on real semisimple Lie algebras that is required for computing $\chi(M)$.

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NOTATION. Throughout this paper, a *hat* means that we are considering some *complexification* of the corresponding bare object: For instance we use $\hat{\mathfrak{g}}$ for the complexification $\boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{g}$ of the real Lie algebra $\mathfrak{g}$, or $\hat{M}$ for the complex flag manifold that contains the minimal orbit M. For the labels of real simple Lie algebras and Lie groups we follow [Hel78, Table VI, Chapter X]. For the labels of the roots and the description of the root systems we refer to [Bou68].

2. Complex flag manifolds. A *complex flag manifold* is the quotient $\hat{M} = \hat{\mathbf{G}}/\mathbf{Q}$ of a complex semisimple Lie group $\hat{\mathbf{G}}$ by a parabolic subgroup $\mathbf{Q}$. We recall that $\mathbf{Q}$ is parabolic in $\hat{\mathbf{G}}$ if and only if its Lie algebra $\mathfrak{q}$ contains a Borel subalgebra, i.e., a maximal solvable subalgebra, of the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{\mathbf{G}}$. We also note that $\hat{\mathbf{G}}$ is necessarily a linear group, and that $\mathbf{Q}$ is connected, contains the center of $\hat{\mathbf{G}}$ and equals the normalizer of $\mathfrak{q}$ in $\hat{\mathbf{G}}$:

$$\mathbf{Q} = \{g \in \hat{\mathbf{G}} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(\mathfrak{q}) = \mathfrak{q}\}. \tag{2.1}$$

In particular, a different choice of a connected $\hat{\mathbf{G}}'$ and of a parabolic $\mathbf{Q}'$, with Lie algebras $\hat{\mathfrak{g}}'$ and $\mathfrak{q}'$ isomorphic to $\hat{\mathfrak{g}}$ and $\mathfrak{q}$, yields a complex flag manifold $\hat{M}'$ that is complex-projectively isomorphic to $\hat{M}$. Thus a flag manifold $\hat{M}$ is better described in terms of the pair of Lie algebras $\hat{\mathfrak{g}}$ and $\mathfrak{q}$.

Fix a Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ that is contained in $\mathfrak{q}$. Let $\mathcal{R}$ be the root system with respect to $\hat{\mathfrak{h}}$ and denote by $\hat{\mathfrak{g}}^{\alpha} = \{Z \in \hat{\mathfrak{g}} \mid [H, Z] = \alpha(H)Z \text{ for any } H \in \hat{\mathfrak{h}}\}$ the root subspace of $\alpha \in \mathcal{R}$. Then we can choose a lexicographic order "$\prec$" of $\mathcal{R}$ such that $\hat{\mathfrak{g}}^{\alpha} \subset \mathfrak{q}$ for all positive α. Let $\mathcal{B}$ be the corresponding system of positive simple roots. All $\alpha \in \mathcal{R}$ are linear combinations of elements of the basis $\mathcal{B}$:

$$\alpha = \sum_{\beta \in \mathcal{B}} k_{\alpha}^{\beta} \beta, \quad k_{\alpha}^{\beta} \in \mathbf{Z} \tag{2.2}$$

and we define the support $\mathrm{supp}_{\mathcal{B}}(\alpha)$ of α with respect to $\mathcal{B}$ as the set of $\beta \in \mathcal{B}$ for which $k_{\alpha}^{\beta} \neq 0$. The set $\mathcal{Q} = \{\alpha \in \mathcal{R} \mid \hat{\mathfrak{g}}^{\alpha} \subset \mathfrak{q}\}$ is a *parabolic set*, i.e., is closed under root addition and $\mathcal{Q} \cup (-\mathcal{Q}) = \mathcal{R}$. Let $\Phi \subset \mathcal{B}$ be the subset of simple roots α for which $\hat{\mathfrak{g}}^{-\alpha} \not\subset \mathfrak{q}$. Then $\mathcal{Q}$ and $\mathfrak{q}$ are completely determined by Φ. Indeed,

$$\mathcal{Q} = \mathcal{Q}_{\Phi} := \{\alpha \succ 0\} \cup \{\alpha \prec 0 \mid \mathrm{supp}_{\mathcal{B}}(\alpha) \cap \Phi = \emptyset\} = \mathcal{Q}_{\Phi}^{r} \cup \mathcal{Q}_{\Phi}^{n}, \tag{2.3}$$

where

$$\mathcal{Q}^r_\Phi = \{\alpha \in \mathcal{R} \mid \mathrm{supp}_{\mathcal{B}}(\alpha) \cap \Phi = \emptyset\} \tag{2.4}$$

$$\mathcal{Q}^n_\Phi = \{\alpha \in \mathcal{R} \mid \alpha \succ 0 \text{ and } \mathrm{supp}_{\mathcal{B}}(\alpha) \cap \Phi \neq \emptyset\}, \tag{2.5}$$

and for the parabolic subalgebra $\mathfrak{q}$ we have the decomposition:

$$\mathfrak{q} = \mathfrak{q}_\Phi = \hat{\mathfrak{h}} + \sum_{\alpha \in \mathcal{Q}_\Phi} \hat{\mathfrak{g}}^\alpha = \mathfrak{q}^r_\Phi \oplus \mathfrak{q}^n_\Phi, \tag{2.6}$$

where

$$\mathfrak{q}^n_\Phi = \sum_{\alpha \in \mathcal{Q}^n_\Phi} \hat{\mathfrak{g}}^\alpha \quad \text{is the nilradical of } \mathfrak{q}_\Phi \text{, and} \tag{2.7}$$

$$\mathfrak{q}^r_\Phi = \hat{\mathfrak{h}} + \sum_{\alpha \in \mathcal{Q}^r_\Phi} \hat{\mathfrak{g}}^\alpha \quad \text{is a reductive complement of } \mathfrak{q}^n_\Phi \text{ in } \mathfrak{q}_\Phi. \tag{2.8}$$

We also set

$$\hat{\mathfrak{h}}'_\Phi = \hat{\mathfrak{h}} \cap [\mathfrak{q}^r_\Phi, \mathfrak{q}^r_\Phi], \tag{2.9}$$

$$\hat{\mathfrak{h}}''_\Phi = \{H \in \hat{\mathfrak{h}} \mid [H, \mathfrak{q}^r_\Phi] = 0\}. \tag{2.10}$$

Then

$$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}'_\Phi \oplus \hat{\mathfrak{h}}''_\Phi \tag{2.11}$$

and $\hat{\mathfrak{h}}''_\Phi$ is the center of the reductive Lie subalgebra $\mathfrak{q}^r_\Phi$.

All Cartan subalgebras of $\hat{\mathfrak{g}}$ are equivalent, modulo inner automorphisms, and all simple basis of a fixed root system $\mathcal{R}$ are equivalent for the transpose of inner automorphisms of $\hat{\mathfrak{g}}$ normalizing $\hat{\mathfrak{h}}$. Thus the correspondence $\Phi \leftrightarrow \mathfrak{q}_\Phi$ is one-to-one between the subsets Φ of an assigned system $\mathcal{B}$ of simple roots of $\mathcal{R}$ and the complex parabolic Lie subalgebras of $\hat{\mathfrak{g}}$, modulo inner automorphisms. In other words, the flag manifolds associated to a connected semisimple complex Lie group with Lie algebra $\hat{\mathfrak{g}}$ are parametrized by the subsets Φ of a basis $\mathcal{B}$ of simple roots of its root system $\mathcal{R}$, relative to any Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$.

The choice of a Cartan subalgebra $\hat{\mathfrak{h}}$ of $\hat{\mathfrak{g}}$ contained in $\mathfrak{q}$ yields a canonical Chevalley decomposition of the parabolic subgroup $\mathbf{Q}$:

PROPOSITION 2.1. *With the notation above, we have a Chevalley decomposition*

$$\mathbf{Q} = \mathbf{Q}^n_\Phi \ltimes \mathbf{Q}^r_\Phi, \tag{2.12}$$

where the unipotent radical $\mathbf{Q}^n_\Phi$ is the connected and simply connected Lie subgroup of $\hat{\mathbf{G}}$ with Lie algebra $\mathfrak{q}^n_\Phi$, and $\mathbf{Q}^r_\Phi$ is the reductive[1] complement with Lie algebra $\mathfrak{q}^r_\Phi$. The reductive $\mathbf{Q}^r_\Phi$ is characterized by

$$\mathbf{Q}^r_\Phi = \mathbf{Z}_{\hat{\mathbf{G}}}(\hat{\mathfrak{h}}''_\Phi) = \{g \in \hat{\mathbf{G}} \mid \mathrm{Ad}_{\hat{\mathfrak{g}}}(g)(H) = H \textit{ for all } H \in \hat{\mathfrak{h}}''_\Phi\}. \tag{2.13}$$

[1] According to [Kna02] we call reductive a linear Lie group $\mathbf{G}$, having finitely many connected components, with a reductive Lie algebra $\mathfrak{g}$, and such that $\mathrm{Ad}_{\hat{\mathfrak{g}}}(\mathbf{G}) \subset \mathrm{Int}(\hat{\mathfrak{g}})$.