A CHARACTERIZATION OF CR QUADRICS WITH A SYMMETRY PROPERTY

ANDREA ALTOMANI AND COSTANTINO MEDORI

Abstract. We study CR quadrics satisfying a symmetry property $(\tilde{S})$ which is slightly weaker than the symmetry property (S), recently introduced by W. Kaup, which requires the existence of an automorphism reversing the gradation of the Lie algebra of infinitesimal automorphisms of the quadric.

We characterize quadrics satisfying the $(\tilde{S})$ property in terms of their Levi-Tanaka algebras. In many cases the $(\tilde{S})$ property implies the (S) property; this holds in particular for compact quadrics.

We also give a negative answer to a question by V. Ezhov and G. Schmalz about the dimension of the algebra of positive-degree infinitesimal automorphisms of a CR quadric.

1. Introduction

The simplest nontrivial example of CR manifolds is given by (affine) CR quadrics, that is submanifolds $Q \subset \mathbb{C}^n_z \times \mathbb{C}^k_w$ of the form $Q = \{(z, w) \mid \Im w = H(z, z)\}$, for an hermitian form $H \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^k$.

The affine quadric Q admits a canonical completion $\hat{Q}$, which is a (in general nonclosed) real subvariety of a complex projective space $\mathbb{CP}^N$, and any local automorphism of Q extends to a global linear automorphism of $\mathbb{CP}^N$ preserving $\hat{Q}$.

The Lie algebra $\mathfrak{g}$ of infinitesimal CR automorphisms of Q (or $\hat{Q}$, as they are locally isomorphic) at $0 \in \mathbb{C}^n \times \mathbb{C}^k$ is naturally graded:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

This algebra is naturally isomorphic to the Levi-Tanaka algebra associated to the quadric, and to the Lie algebra of the group $\mathrm{Aut}_{\mathrm{CR}}(\hat{Q})$ of CR automorphisms of $\hat{Q}$. Then $\mathrm{Aut}_{\mathrm{CR}}(\hat{Q})$ acts on $\mathfrak{g}$ via the adjoint action.

In [7] W. Kaup introduced the following definition: A quadric Q has the symmetry property, which we call (S), if there exists an involutive automorphism γ of $\hat{Q}$ such that $\mathrm{Ad}(\gamma)(\mathfrak{g}_j) = \mathfrak{g}_{-j}$ for $j = -2, \ldots, 2$.

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In this paper we consider a slight generalization of property (S), that we call property $(\tilde{S})$, as we only require the existence of a degree-reversing automorphism of $\hat{Q}$ of finite order. We then characterize CR quadrics with property $(\tilde{S})$ in terms of the gradation: a quadric has property $(\tilde{S})$ if and only if the gradation of $\mathfrak{g}$ is induced by the adjoint action of an element of a semisimple Levi factor of $\mathfrak{g}$. Moreover, the element γ can always be chosen of order 2 or 4. In several cases, including all compact quadrics, we show that it is possible to choose γ of order 2, thus recovering property (S).

Finally we present an example of a CR quadric such that $\dim \mathfrak{g}_1 + \dim \mathfrak{g}_2 > \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-2}$, giving a negative answer to a question of V. Ezhov and G. Schmalz [6], who asked if the inequality $\dim \mathfrak{g}_1 + \dim \mathfrak{g}_2 \leq \dim \mathfrak{g}_{-1} + \dim \mathfrak{g}_{-2}$ always holds true.

2. CR manifolds and Levi-Tanaka algebras

We first recall the definition of a CR manifold.

Definition 2.1. A CR *manifold* of type (n,k) is the datum $(M, T^{1,0}M)$ of a real smooth manifold M of dimension $2n+k$ and a smooth complex vector subbundle $T^{1,0}M$, with constant complex rank n, of the complexification $T^{\mathbb{C}}M$ of the tangent bundle of M, satisfying the following conditions:

(1) $T^{1,0}M \cap \overline{T}^{1,0}M = 0$,
(2) $[C^\infty(M, T^{1,0}M), C^\infty(M, T^{1,0}M)] \subset C^\infty(M, T^{1,0}M)$.

The integers n and k are called the CR *dimension* and CR *codimension* of M. We also set $T^{0,1}M = \overline{T^{1,0}M}$ and $HM = TM \cap (T^{1,0}M + T^{0,1}M)$.

Let $J\colon T^{1,0}M + T^{0,1}M \to T^{1,0}M + T^{0,1}M$ be the linear semisimple isomorphism with eigenvalues i on $T^{1,0}M$ and $-$i on $T^{0,1}M$. Then J preserves HM and is called a *partial complex structure*.

A CR *map* between two CR manifolds M and N is a smooth map $f\colon M \to N$ such that $df^{\mathbb{C}}(T^{1,0}M) \subset T^{1,0}N$. The notions of CR isomorphism and automorphism are defined in the natural way.

Definition 2.2. Let M be a real submanifold of a complex manifold X. Define $T^{1,0}M = T^{\mathbb{C}}M \cap T^{1,0}X$. If $T^{1,0}M$ has constant rank, then $(M, T^{1,0}M)$ is a CR manifold, called a CR *submanifold* of X.

We introduce two further definitions

Definition 2.3. A CR manifold $(M, T^{1,0}M)$ is said to be:

(1) *Levi nondegenerate* at a point $x \in M$ if for every vector field $Z \in C^\infty(M, T^{1,0}M)$ there exists a vector field $\bar{W} \in C^\infty(M, T^{0,1}M)$ such that $[Z, \bar{W}]_x \notin T^{1,0}_xM + T^{0,1}_xM$;
(2) of *finite type* at a point $x \in M$ if the Lie algebra generated by all vector fields in $C^\infty(M, T^{1,0}M + T^{0,1}M)$ spans $T^{\mathbb{C}}_xM$.

2.1. **Levi-Tanaka algebras and standard CR manifolds.** Let M be a CR manifold, and $x \in M$ a point where M is Levi nondegenerate and of finite type. We associate to x a graded Lie algebra, called the Levi-Tanaka algebra of M at x. We refer to [8] for a more detailed discussion of Levi-Tanaka algebras. Define:

$$\mathcal{D}_0 = 0, \qquad \mathcal{D}_{-1} = C^\infty(M, HM),$$

and inductively, for $p \geq 2$:

$$\mathcal{D}_{-p} = \mathcal{D}_{-p+1} + [\mathcal{D}_{-p+1}, \mathcal{D}_{-1}].$$

We define then, for $p \geq 1$:

$$\mathfrak{m}_{-p} = \mathcal{D}_{-p}(x)/\mathcal{D}_{-p+1}(x).$$

The vector field bracket induces a graded Lie algebra structure on $\mathfrak{m}_- = \sum_{p\leq -1} \mathfrak{m}_p$. Note that $\mathfrak{m}_{-1}$ is canonically isomorphic to H_xM. Then it is naturally defined a complex structure J on $\mathfrak{m}_{-1}$ and $[JX, JY] = [X, Y]$ for every $X, Y \in \mathfrak{m}_{-1}$.

Let

$$\mathfrak{m}_0 = \{D \in \mathrm{Der}_0(\mathfrak{m}_-) \mid [D|_{\mathfrak{m}_{-1}}, J] = 0\}$$

be the set of zero-degree derivations on $\mathfrak{m}_-$ commuting with J on $\mathfrak{m}_{-1}$. Then $\mathfrak{m}_0 + \mathfrak{m}_-$ is a graded Lie algebra.

Definition 2.4. The *Levi-Tanaka algebra* associated to M at a point $x \in M$ where M is Levi nondegenerate and of finite type is the unique graded Lie algebra $\mathfrak{g} = \sum_{p\in\mathbb{Z}} \mathfrak{g}_p$ with the following properties:

(1) $\mathfrak{g}_p = \mathfrak{m}_p$ for $p \leq 0$,
(2) for all $X \in \mathfrak{g}_p$, with $p \geq 0$, the action $\mathrm{ad}_\mathfrak{g}(X)|_{\mathfrak{g}_{-1}}$ is nonzero,
(3) $\mathfrak{g}$ is maximal with those properties.

Definition 2.5. In general, we can start with any graded Lie algebra $\mathfrak{m}_- = \sum_{p\leq -1} \mathfrak{m}_p$, such that $\mathfrak{m}_{-1}$ generates $\mathfrak{m}_-$ and with a complex structure J on $\mathfrak{m}_{-1}$ such that

$$[X, Y] = [JX, JY] \qquad X, Y \in \mathfrak{m}_{-1},$$

and perform the same prolongation procedure as in Definition 2.4.

The resulting algebra $\mathfrak{g} = \sum_{p\in\mathbb{Z}} \mathfrak{g}_p$ (with complex structure J on $\mathfrak{g}_{-1}$) is a *Levi-Tanaka algebra*.

We fix the following notation: for a graded Lie algebra $\mathfrak{g} = \sum_{p\in\mathbb{Z}} \mathfrak{g}_p$, we set

$$\mathfrak{g}_- = \sum_{p<0} \mathfrak{g}_p, \qquad \mathfrak{p} = \sum_{p\geq 0} \mathfrak{g}_p,$$

$$\mathfrak{g}_+ = \sum_{p>0} \mathfrak{g}_p, \quad \mathfrak{p}^{\mathrm{opp}} = \sum_{p\leq 0} \mathfrak{g}_p.$$

A Levi-Tanaka algebra has trivial center and contains a unique element $E \in \mathfrak{g}_0$, called *characteristic element*, such that $\mathrm{ad}_\mathfrak{g}(E)|_{\mathfrak{g}_j} = j\,\mathrm{Id}_{\mathfrak{g}_j}$ for all $j \in \mathbb{Z}$.

Definition 2.6. For a Levi-Tanaka algebra $\mathfrak{g} = \sum_{p\in\mathbb{Z}} \mathfrak{g}_p$, with complex structure $J_\mathfrak{g}$ on $\mathfrak{g}_{-1}$, it is possible to construct a CR manifold such that the associated Levi-Tanaka algebra at every point is isomorphic to $\mathfrak{g}$.

Let $\tilde{\mathbf{G}}$ be the connected and simply connected group with Lie algebra $\mathfrak{g}$, and $\mathbf{P}$ the analytic subgroup with Lie algebra $\mathfrak{p}$. Then $\mathbf{P}$ is closed, and we let $S = S(\mathfrak{g}) = \tilde{\mathbf{G}}/\mathbf{P}$.

There is a unique $\tilde{\mathbf{G}}$-homogeneous CR structure on S such that at the base point $o = e\mathbf{P}$ the partial complex structure is given by $H_oS = \mathfrak{g}_{-1}$ and $J_o = J_\mathfrak{g}$, where we identified in the natural way T_oS and $\mathfrak{g}_-$.

The CR manifold $S(\mathfrak{g})$ is the *standard* CR *manifold* associated to $\mathfrak{g}$.

The standard CR manifold S is simply connected, and the group of CR automorphisms of S has Lie algebra $\mathfrak{g}$. This group is in general not connected, but we can identify its connected component of the identity.

Proposition 2.7. *Let $\mathfrak{g}$ be a Levi-Tanaka algebra, and S the associated standard* CR *manifold. Then the connected component of the identity of the group* $\mathrm{Aut}_{\mathrm{CR}}(S)$ *of* CR *automorphisms of S is the group* $\mathbf{G}^0 = \mathrm{Int}(\mathfrak{g})$ *of inner automorphisms of* $\mathfrak{g}$.

Proof. Let $\mathbf{P}^0 = \{g \in \mathbf{G}^0 \mid \mathrm{Ad}(g)(\mathfrak{p}) = \mathfrak{p}\}$. Then the Lie algebra of $\mathbf{P}^0$ is $\mathfrak{p}$ and the manifold $M = \mathbf{G}^0/\mathbf{P}^0$ has a natural CR structure. The natural quotient $\tilde{\mathbf{G}} \to \mathbf{G}^0$ induces a covering map $\pi\colon S \to M$.

For every $X \in \mathfrak{g}_-$ we have $\mathrm{ad}(X)(E) \in \mathfrak{g}_-$, hence $\exp_{\mathbf{G}^0}(\mathfrak{g}_-) \cap \mathbf{P}^0 = \{e\}$. Since the orbit of $\exp_{\tilde{\mathbf{G}}}(\mathfrak{g}_-) \cdot o$ is dense in S (see [10]), the map π is one-to-one, and hence a diffeomorphism. □

The standard CR manifold S can then be identified to the set of inner conjugates of $\mathfrak{p}$ in $\mathfrak{g}$.

This observation provides also another construction of standard CR manifolds. The group $\mathbf{G}^0$ acts, via the complexification of the adjoint action, on all the complex grassmannians of subspaces of $\mathfrak{g}^\mathbb{C}$. Let

$$\mathfrak{q} = \mathfrak{g}_2^\mathbb{C} + \mathfrak{g}_1^\mathbb{C} + \mathfrak{g}_0^\mathbb{C} + \{X + \mathrm{i}JX \mid X \in \mathfrak{g}_{-1}\}.$$

The $\mathbf{G}^0$-orbit through the point $o = \mathfrak{q}$ in the complex grassmannian $\mathrm{Gr}_{\dim \mathfrak{q}}(\mathfrak{g}^\mathbb{C})$, with the CR structure given by the embedding, is CR-isomorphic to the standard CR manifold associated to $\mathfrak{g}$.

Although we will not use it, we give a characterization of the full automorphism group of a standard CR manifold.

Proposition 2.8. *Let $\mathfrak{g}$ be a Levi-Tanaka algebra, and S the associated standard* CR *manifold. Then the group* $\mathrm{Aut}_{\mathrm{CR}}(S)$ *of* CR *automorphisms of S is the group*

$$\mathbf{G} = \{g \in \mathrm{Aut}(\mathfrak{g}) \mid g \cdot \mathfrak{q} \text{ is } \mathrm{Int}(\mathfrak{g})\text{-conjugate to } \mathfrak{q}\}.$$

Proof. ($\mathrm{Aut}_{\mathrm{CR}}(S) \subset \mathbf{G}$). Let $\phi \in \mathrm{Aut}_{\mathrm{CR}}(S)$ be a CR automorphism of S and $X \in \mathfrak{g}$. Denote by $X^\dagger$ the vector field on S generated by X. Then $(\phi \cdot X)^\dagger = d\phi(X^\dagger)$ defines an action of ϕ on $\mathfrak{g}$, which is an automorphism. Let $g \in \mathrm{Int}(\mathfrak{g})$ be an element such that $\phi \circ g(o) = o$. Then $\phi \circ g \cdot \mathfrak{p} = \mathfrak{p}$, and $\phi \circ g \cdot \mathfrak{q} = \mathfrak{q}$ because $\phi \circ g$ is a CR map.

($\mathrm{Aut}_{\mathrm{CR}}(S) \supset \mathbf{G}$). Let g be an element of $\mathbf{G}$, and $h \in \mathrm{Int}(\mathfrak{g})$ an element with $g \cdot \mathfrak{q} = h \cdot \mathfrak{q}$. Define an action of g on S as follows: for $k \in \mathrm{Int}(\mathfrak{g})$, let $g \cdot (k \cdot o) = (gkhg^{-1}) \cdot o$. □

3. CR quadrics

Let $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^k$ be a vector valued hermitian form, linear in the first variable and anti-$\mathbb{C}$-linear in the second one.

Definition 3.1. The vector valued hermitian form $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^k$ is said to be:

nondegenerate: if for all $z \in \mathbb{C}^n \setminus \{0\}$ there exists $z' \in \mathbb{C}^n$ such that $H(z, z') \neq 0$;

fundamental: if the set $\{H(z, z) \mid z \in \mathbb{C}^n\} \subset \mathbb{R}^k$ spans $\mathbb{R}^k$.

To a vector valued hermitian form it is naturally associated a CR submanifold of $\mathbb{C}^{n+k}$.

Definition 3.2. The *affine* CR *quadric associated to* a vector valued hermitian form $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^k$ is the CR-submanifold of $\mathbb{C}^n \oplus \mathbb{C}^k$ given by:

$$Q = Q^H = \{(z, w) \in \mathbb{C}^n \oplus \mathbb{C}^k \mid \Im w = H(z, z)\}.$$

It is straightforward to see that Q^H is a CR manifold of CR-dimension n and CR-codimension k, it is finitely nondegenerate (in fact Levi nondegenerate) if and only if H is nondegenerate, and it is of finite type (indeed of type 2) if and only if H is fundamental.

Remark 3.3. Any affine quadric Q can be written as a product $Q = Q' \times \mathbb{C}^m \times \mathbb{R}^h$, where Q' is a Levi nondegenerate affine quadric of finite type, m is the dimension of the null space of H, and h is the codimension of the image of H in $\mathbb{C}^k$.

We assume, from now on, that H is nondegenerate and fundamental.

The Lie algebra of infinitesimal automorphisms of Q is finite-dimensional and possesses a natural grading $\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}_i$. It is canonically isomorphic to the Levi-Tanaka algebra associated to Q (see [13] and [5]).

Let $(\mathfrak{g} = \oplus_{i \in \mathbb{Z}} \mathfrak{g}_i, J)$ be the Levi-Tanaka algebra associated to Q. Then $\mathfrak{g}_i = 0$ for $i < -2$ and $i > 2$, and $\dim_{\mathbb{R}} \mathfrak{g}_{-1} = 2n$, $\dim_{\mathbb{R}} \mathfrak{g}_{-2} = k$. The Lie algebra structure on $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}$ is given by H in the following way. Identify $\mathfrak{g}_{-1}$ with its complex structure J to $\mathbb{C}^n$, and $\mathfrak{g}_{-2}$ to $\mathbb{R}^k \subset \mathbb{C}^k$. Then:

$$[X, Y] = \Im(H(X, Y)), \qquad \forall X, Y \in \mathfrak{g}_{-1}.$$

Definition 3.4. The *quadric* $\hat{Q} = \hat{Q}^H$ associated to H is the standard CR manifold $S(\mathfrak{g})$ associated to $\mathfrak{g}$. This definition agrees with the definition in [7].

The affine quadric Q is CR diffeomorphic to the $\mathbf{G}_-$ orbit through o, and it is open and dense in $\hat{Q}$. The complement $\hat{Q} \setminus Q$ is the intersection of $\hat{Q}$ and a complex-algebraic subvariety of $\mathrm{Gr}_{\dim \mathfrak{q}}(\mathfrak{g}^{\mathbb{C}})$. The isotropy Lie algebra at o is $\mathfrak{p}$, and the isotropy subgroup is $\mathbf{P}^0$.

In [7] W. Kaup introduced a symmetry property, called property (S), for the quadric $\hat{Q}$. Here we consider the following generalization.

Definition 3.5. The quadric $\hat{Q}$ is said to have:

- *the* (S) *property* if there exists an involutive automorphism $\gamma \in \mathbf{G}$ such that $\mathrm{Ad}_{\mathfrak{g}}(\gamma)(E) = -E$;
- *the* $(\tilde{S})$ *property* if there exists an automorphism $\gamma \in \mathbf{G}$ of finite order such that $\mathrm{Ad}_{\mathfrak{g}}(\gamma)(E) = -E$.

We use the same notation for the Levi-Tanaka algebra associated to the quadric.

Our aim is to characterize quadrics with the $(\tilde{S})$ property, and show that in many cases the (S) and $(\tilde{S})$ properties are equivalent.

4. Levi-Malcev decomposition

We recall that Levi Tanaka algebras have a *pseudocomplex graded* Levi-Malcev decomposition, i.e. compatible with the grading and the complex structure [11]. More precisely, given a Levi-Tanaka algebras $\mathfrak{g}$, with radical $\mathfrak{r}$, there exist a semisimple subalgebra $\mathfrak{s}$ such that:

(1) $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$;
(2) $\mathfrak{s}$ and $\mathfrak{r}$ are graded;
(3) $\mathfrak{s}_{-1}$ and $\mathfrak{r}_{-1}$ are J-invariant.

Lemma 4.1. *Let* $\mathfrak{g} = \sum_{p \in \mathbb{Z}} \mathfrak{g}_p$ *be a finite dimensional Levi-Tanaka algebra, and* $\Gamma \in \mathrm{Aut}(\mathfrak{g})$ *an automorphism with* $\Gamma(E) = -E$. *Let* $\mathfrak{r} = \sum \mathfrak{r}_p$ *be radical of* $\mathfrak{g}$. *Then* $\mathfrak{r}_{-2} := \mathfrak{r} \cap \mathfrak{g}_{-2} \neq \mathfrak{g}_{-2}$.

Proof. Assume $\mathfrak{r} \cap \mathfrak{g}_{-2} = \mathfrak{g}_{-2}$. Then we have $\mathfrak{r} \cap \mathfrak{g}_p = \mathfrak{g}_p$ for every $p \leq -2$. Let $\mathfrak{n} = \sum \mathfrak{n}_p$ be the nilradical of $\mathfrak{g}$ and consider the descending central sequence

$$\mathfrak{n}^1 = \mathfrak{n}, \qquad \mathfrak{n}^{k+1} = [\mathfrak{n}^k, \mathfrak{n}], \text{ for } k \geq 1.$$

Note that $\sum_{p \neq 0} \mathfrak{r}_p \subset \mathfrak{n}$. Let d be the minimal integer such that $\mathfrak{n}^d \neq 0$ and $\mathfrak{n}^{d+1} = 0$. Then $\mathfrak{n}^d$ is a characteristic ideal of $\mathfrak{g}$ and $[\mathfrak{n}, \mathfrak{n}^d] = 0$.

Consider $X \in \mathfrak{n}^d_1 := \mathfrak{n}^d \cap \mathfrak{g}_1$. We have

$$[X, \mathfrak{g}_p] = [X, \mathfrak{r}_p] = [X, \mathfrak{n}_p] = 0\,, \quad \forall p \leq -2, \tag{4.1}$$

hence $X = 0$ (see [8, Theorem 3.1]). Then $\mathfrak{n}^d_1 = 0$ and in general $\mathfrak{n}^d_p := \mathfrak{n}^d \cap \mathfrak{g}_p = 0$, for any $p > 0$.

Since Γ interchanges $\mathfrak{g}_p$ and $\mathfrak{g}_{-p}$, and the ideal $\mathfrak{n}^d$ is characteristic, we have also $\mathfrak{n}^d_p = 0$ for $p \neq 0$, therefore $\mathfrak{n}^d \subset \mathfrak{g}_0$.

Finally,

$$[\mathfrak{n}^d_0, \mathfrak{g}_{-1}] \subset \mathfrak{n}^d_{-1} = \{0\},$$

hence $\mathfrak{n}^d_0 = \{0\}$. Then we have $\mathfrak{n}^d = \{0\}$, obtaining a contradiction. □

We fix now a pseudocomplex graded Levi-Malcev decomposition

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$$

of the Levi-Tanaka algebra $\mathfrak{g}$ associated to a quadric $\hat{Q}$ having the $(\tilde{S})$ property. From Lemma 4.1 it follows that $\mathfrak{s}_2 \neq 0$ and $\mathfrak{s}_{-2} \neq 0$.

Let $E \in \mathfrak{g}$ be the characteristic element. Then $E = E_{\mathfrak{s}} + E_{\mathfrak{r}}$ with $E_{\mathfrak{r}} \in \mathfrak{r}$, and $E_{\mathfrak{s}} \in \mathfrak{s}$ is the characteristic element of $\mathfrak{s}$.

Proposition 4.2. *If a quadric $\hat{Q}$ admits an automorphism γ with $\mathrm{Ad}(\gamma)(E) = -E$, then $E = E_{\mathfrak{s}}$.*

Proof. Assume that $\hat{Q}$ admits such a γ. The isotropy Lie algebra at the point $\gamma \cdot o \in \hat{Q}$ is $\mathfrak{g}_0 \oplus \mathfrak{g}_-$. It follows that there exists an element $X_+ \in \mathfrak{g}_+$ with $\exp(X_+)\gamma \cdot o \in Q$. Since $\exp(\mathfrak{g}_-)$ acts transitively on Q, we also have an element $X_- \in \mathfrak{g}_-$ such that $\exp(X_+)\gamma \cdot o = \exp(X_-) \cdot o$ or in other words: $\gamma = \exp(-X_+)\exp(X_-)h$, where h is an element of the isotropy at o. Since the isotropy at o is exactly $\mathbf{G}_0\mathbf{G}_+$, we finally obtain, for a $g_0 \in \mathbf{G}_0$ and an $X'_+ \in \mathbf{G}_+$:

$$\begin{aligned}\gamma &= \exp(-X_+)\exp(X_-)\exp(X'_+)g_0 \\ &= \exp(Y^1_1)\exp(Y^1_2)\exp(Y^2_{-1})\exp(Y^2_{-2})\exp(Y^3_1)\exp(Y^3_2)g_0\end{aligned}$$

(here the subscripts indicate the degrees of the homogeneous elements Y^i_j).

From $\mathrm{Ad}(\gamma)(E) = -E$ we obtain

$$2E = 2[Y^2_{-2}, Y^3_2] + \frac{1}{2}[Y^2_{-1}, Y^1_1 + Y^3_1].$$

Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$. Note that it is graded, and $\mathfrak{r}_p = \mathfrak{n}_p$ for all $p \neq 0$. Decompose each element Y^i_j into its $\mathfrak{s}$ and $\mathfrak{n}$ component. It follows

$$2E_{\mathfrak{r}} \in ([\mathfrak{s}, \mathfrak{n}] + [\mathfrak{r}, \mathfrak{r}]) \cap \mathfrak{g}_0 \subset \mathfrak{n}_0$$

and $E_{\mathfrak{r}}$ is ad-nilpotent.

Since $\mathrm{ad}(E)$ preserves $\mathfrak{s}$, and $\mathfrak{r}$ is an ideal, we have $\mathrm{ad}(E)|_{\mathfrak{s}} = \mathrm{ad}(E_{\mathfrak{s}})|_{\mathfrak{s}}$, and $E_{\mathfrak{s}}$ is a ad-semisimple element of $\mathfrak{s}$. Then $E = E_{\mathfrak{s}} + E_{\mathfrak{r}}$ is a Wedderburn decomposition of E, and since E is semisimple element, it follows that $E = E_{\mathfrak{s}}$. □

We also have the following

Lemma 4.3. *If the quadric $\hat{Q}$ has property $(\tilde{S})$, then $\mathfrak{p}^{\mathrm{opp}}$ is conjugate to $\mathfrak{p}$ by an inner automorphism of $\mathfrak{g}$.*

Proof. Assume that $\hat{Q}$ has property $(\tilde{S})$. Since $\mathrm{Int}(\mathfrak{g})$ acts transitively on $\hat{Q}$ and $\mathrm{Ad}(\gamma)(\mathfrak{p}) = \mathfrak{p}^{\mathrm{opp}}$ is the isotropy Lie algebra at the point $\gamma \cdot o \in \hat{Q}$, the condition is necessary. □

5. The semisimple case

We assume now that $\mathfrak{g}$ is semisimple. For a standard CR manifold (and in particular for quadrics) this is equivalent to compactness. First we recall the description of semisimple Levi-Tanaka algebras (see [9] for a more detailed treatment of the topic).

Let $\mathfrak{g}$ be a semisimple Levi-Tanaka algebra. Since every semisimple Levi-Tanaka algebra is a direct sum of simple Levi-Tanaka algebras, we can assume that $\mathfrak{g}$ is simple. Choose a maximally noncompact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, let $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$, and $\mathcal{R} = \mathcal{R}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ be the corresponding root system.

The conjugation of $\mathfrak{g}^{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$ leaves $\mathfrak{h}^{\mathbb{C}}$ invariant, and then induces a conjugation $\sigma\colon \mathcal{R} \to \mathcal{R}$. Let $\mathcal{R}^\bullet = \{\alpha \in \mathcal{R} \mid \sigma\alpha = -\alpha\}$ be the set of compact roots. There exists a choice of a set of positive roots $\mathcal{R}_+$ such that $\sigma(\mathcal{R}_+ \setminus \mathcal{R}^\bullet) \subset \mathcal{R}_+$. Let $\mathcal{B}$ be the system of simple positive roots for $\mathcal{R}_+$, that we identify to the nodes of the associated Dynkin diagram Δ, and $\mathcal{B}^\bullet = \mathcal{B} \cap \mathcal{R}^\bullet$.

The action of the conjugation σ on simple positive roots can be described as follows: there exists an involution of the Dynkin diagram $\epsilon\colon \mathcal{B} \to \mathcal{B}$ such that $\sigma\alpha - \epsilon\alpha \in \langle \mathcal{B}^\bullet \rangle_{\mathbb{Z}}$. The datum of $(\Delta, \mathcal{B}^\bullet, \epsilon)$ completely determines $\mathfrak{g}$ and is known as Satake diagram of $\mathfrak{g}$.

Fix a subset Φ of $\mathcal{B}$ with the following properties:

(1) $\Phi \cap \mathcal{B}^\bullet = \emptyset$,
(2) $\Phi \cap \epsilon\Phi = \emptyset$ (in particular ϵ is nontrivial),
(3) every connected component of Δ intersects both Φ and $\epsilon\Phi$,
(4) every connected segment in Δ containing two elements of Φ, also intersects $\epsilon\Phi$.

Let E, J be the elements of $\mathfrak{h}$ such that:

$$\begin{cases} \alpha(E) = 1 & \text{for } \alpha \in \Phi \cup \epsilon(\Phi), \\ \alpha(E) = 0 & \text{for } \alpha \notin \Phi \cup \epsilon(\Phi), \end{cases}$$

$$\begin{cases} \alpha(J) = -\mathrm{i} & \text{for } \alpha \in \Phi, \\ \alpha(J) = \mathrm{i} & \text{for } \alpha \in \epsilon(\Phi), \\ \alpha(J) = 0 & \text{for } \alpha \notin \Phi \cup \epsilon(\Phi). \end{cases}$$

Then E defines a gradation on $\mathfrak{g}$, and J defines a complex structure on $\mathfrak{g}_{-1}$. The largest $p \in \mathcal{N}$ such that $\mathfrak{g}_p \neq \{0\}$ is called the *kind* of $\mathfrak{g}$. It coincides with the degree of a maximal positive root. Conversely, every simple Levi-Tanaka algebra is isomorphic to one obtained in this way.

It is straightforward then to classify the simple Levi-Tanaka algebras of kind 2. The names of the simple Lie algebras of real type are those of the corresponding symmetric spaces in Cartan's classification, the order of the roots $\{\alpha_1, \ldots, \alpha_\ell\} = \mathcal{B}$ follows Bourbaki (see the table in the appendix of [1] or [2]). For simple algebras of the complex type, the simple roots are denoted $\{\alpha_1, \ldots, \alpha_\ell, \alpha'_1, \ldots, \alpha'_\ell\}$ with $\epsilon\alpha_j = \alpha'_j$.

Proposition 5.1. *The simple Levi-Tanaka algebras of kind* 2 *are direct sums of simple factor of the following types:*

1. *Type* $\mathrm{A}_\ell\,\mathrm{III/IV}$*,* $\Phi = \{\alpha_i\}$*, with* $1 \le i \le p$ *and* $i \neq (\ell+1)/2$ *(or* $q \le i \le \ell$ *and* $i \neq (\ell+1)/2$*);*
2. *Type* $\mathrm{D}_\ell\,\mathrm{Ib/IIIb}$*,* $\Phi = \{\alpha_\ell\}$ *(or* $\Phi = \{\alpha_{\ell-1}\}$*);*
3. *Type* $\mathrm{E\,II/III}$*,* $\Phi = \{\alpha_1\}$ *(or* $\Phi = \{\alpha_6\}$*);*
4. *Type* $\mathrm{A}_\ell^{\mathbb{C}}$*,* $\Phi = \{\alpha_i, \alpha'_j\}$ *with* $i \neq j$*;*
5. *Type* $\mathrm{D}_\ell^{\mathbb{C}}$*,* $\Phi = \{\alpha_1, \alpha'_{\ell-1}\}$ *or* $\Phi = \{\alpha_{\ell-1}, \alpha'_\ell\}$ *(or* $\Phi = \{\alpha_1, \alpha'_\ell\}$ *or* $\Phi = \{\alpha_{\ell-1}, \alpha'_1\}$*,* $\Phi = \{\alpha_\ell, \alpha'_{\ell-1}\}$ *or* $\Phi = \{\alpha_\ell, \alpha'_1\}$*);*
6. *Type* $\mathrm{E}_6^{\mathbb{C}}$ *with* $\Phi = \{\alpha_1, \alpha'_6\}$ *(or* $\Phi = \{\alpha_6, \alpha'_1\}$*).* □

We fix then a compact quadric $\hat{Q}$, the corresponding semisimple Levi-Tanaka algebra $\mathfrak{g}$, a maximally noncompact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_0$, a system $\mathcal{B}$ of positive simple roots of the root system $\mathcal{R} = \mathcal{R}(\mathfrak{g}^C, \mathfrak{h}^{\mathbb{C}})$.

Of course in this case $E = E_{\mathfrak{s}}$. We recall that $\mathbf{G}^0 = \mathrm{Int}(\mathfrak{g}) = \mathrm{Aut}_{\mathrm{CR}}(\hat{Q})^0$ is the adjoint group, and $\hat{Q}$ can be identified to the set of $\mathrm{Ad}(\mathbf{G}^0)$-conjugates of $\mathfrak{p}$ in $\mathfrak{g}$ or of $\mathfrak{q}$ in $\mathfrak{g}^{\mathbb{C}}$. We also recall that the *analytic Weyl group* $\mathbf{W}(\mathbf{G}^0, \mathfrak{h})$ is the quotient of the normalizer in $\mathbf{G}^0$ of $\mathfrak{h}$ by the centralizer in $\mathbf{G}^0$ of $\mathfrak{h}$.

First we prove that in the semisimple case the converse of Lemma 4.3 holds true.

Lemma 5.2. *The quadric* $\hat{Q}$ *has property* $(\tilde{S})$ *if and only if* $\mathfrak{p}^{\mathrm{opp}}$ *is conjugate to* $\mathfrak{p}$ *by an inner automorphism of* $\mathfrak{g}$*.*

Proof. If $\mathfrak{p}^{\mathrm{opp}}$ is conjugate by an inner automorphism to $\mathfrak{p}$, we can choose a Weyl group element w with $w \cdot \mathfrak{p} = \mathfrak{p}^{\mathrm{opp}}$. A representative γ of finite order in $\mathbf{G}^0$, which exists thanks to [14], satisfies the $(\tilde{S})$ property. □

5.1. **Simple factors of the real type.** We show that, for simple Lie algebras of the real type, the $(\tilde{S})$ property always holds true. We recall that the analytic Weyl group of a real connected semisimple Lie group $\mathbf{G}^0$ with respect to a real Cartan subalgebra $\mathfrak{h}$ is the group:

$$\mathbf{W}(\mathbf{G}^0, \mathfrak{h}) = \mathbf{N}_{\mathbf{G}^0}(\mathfrak{h}^{\mathbb{C}})/\mathbf{Z}_{\mathbf{G}^0}(\mathfrak{h}^{\mathbb{C}}).$$

It is a subgroup of the usual Weyl group $\mathbf{W}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$.

Lemma 5.3. *If* $\mathfrak{g}$ *is a simple algebra of the real types* A III/IV*,* D Ib*,* D IIIb*,* E II/III *and* $\mathfrak{h}$ *is a maximally split Cartan subalgebra, then the* *longest element w_0 of the Weyl group $\mathbf{W}(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ is in the analytic Weyl group $\mathbf{W}(\mathbf{G}^0, \mathfrak{h})$.*

Proof. First of all, we recall that, for Lie algebras of type D_ℓ and ℓ odd, the longest element w_0 of the Weyl group is minus the identity, while in the other cases of the lemma, w_0 is equal to minus the identity composed with the root involution associated to the symmetry of the Dynkin diagram (see [4]).

If $\mathfrak{g}$ is of type $\mathrm{A\,III/IV}_\ell$, then the roots $\beta_j = \mathrm{e}_j - \mathrm{e}_{\ell+2-j}$, $1 \leq j \leq (\ell+1)/2$, are either real or compact, hence the associated symmetries s_{β_j} are in the analytic Weyl group. The longest element is $w_0 = \Pi_j s_{\beta_j}$.

If $\mathfrak{g}$ is of type $\mathrm{D\,Ib}_\ell$ with $\ell = 2k+1$ odd, or of type $\mathrm{D\,IIIb}_n$, the roots $\mathrm{e}_{2i-1} \pm \mathrm{e}_{2i}$, for $1 \leq i \leq k$, are either real or compact, hence the associated symmetries $s_{\mathrm{e}_{2i-1} \pm \mathrm{e}_{2i}}$ are in the analytic Weyl group, and their product is the longest element w_0.

If $\mathfrak{g}$ is of type $\mathrm{D\,Ib}_\ell$ with $\ell = 2k$ even, the roots $\mathrm{e}_{2i-1} \pm \mathrm{e}_{2i}$, for $1 \leq i \leq k$, are real, hence the associated symmetries $s_{\mathrm{e}_{2i-1} \pm \mathrm{e}_{2i}}$ are in the analytic Weyl group, and furthermore also the symmetry $s_{\mathrm{e}_{2k-1} - \mathrm{e}_{2k}} \circ s_{\mathrm{e}_{2k-1} + \mathrm{e}_{2k}}$ belongs to it. Their product is the longest element w_0.

If $\mathfrak{g}$ is of type $\mathrm{E\,II/III}$, the roots $\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$ and $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ are real, and the roots α_4 and $\alpha_3 + \alpha_4 + \alpha_5$ are either real or compact, hence the associated symmetries are in the analytic Weyl group, and their product is the longest element w_0. □

Proposition 5.4. *If $\mathfrak{g}$ is a simple algebra of the real types* A III/IV, D Ib, D IIIb, E II/III, *then there exists an element of finite order $\gamma \in \mathbf{G}^0$ such that $\mathrm{Ad}(\gamma)(E) = -E$.*

Proof. The longest element w_0 of the Weyl group acts on $\mathfrak{h}$ either by $-\mathrm{Id}$ or by $-\mathrm{Id} \circ \epsilon$, where ϵ is the map induced by the nontrivial automorphism of the diagram. Since E is ϵ-invariant, $w_0 \cdot E = -E$.

Finally, according to [14], there exists a representative γ of w_0 in $\mathbf{G}^0$, of order 2 or 4. □

5.2. **Simple factors of the complex type.** We consider now the case where $\mathfrak{g}$ is a simple algebra of the complex types $A^{\mathbb{C}}$, $D^{\mathbb{C}}$, or $E^{\mathbb{C}}$.

Lemma 5.5. *If a quadric $\hat{Q}$ admits an automorphism of finite order γ with $\mathrm{Ad}(\gamma)(E) = -E$, then there exists a maximally split Cartan subalgebra, containing E, self-conjugate, contained in $\mathfrak{p}$, and $\mathrm{Ad}(\gamma)$-invariant.*

Proof. Let $\Gamma \subset \mathrm{Aut}(\mathfrak{g}^{\mathbb{C}})$ be the group generated by $\mathrm{Ad}(\gamma)$ and complex conjugation. It is a finite group, and it is the direct product of a cyclic group and $\mathbb{Z}/2\mathbb{Z}$. The subalgebra $\mathfrak{g}_0^{\mathbb{C}}$ is Γ-invariant. By [3] there exists a Γ-invariant Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{g}_0^{\mathbb{C}}$. It contains E, because E is in the center. Since $\mathfrak{g}_0^{\mathbb{C}}$ contains a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, also $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Finally, $\mathfrak{h} = \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{g}$ is maximally split because there exists only one conjugacy class of Cartan subalgebras. □

Fix an S-adapted Weyl chamber and system of simple positive roots.

In this case the analytic Weyl group $\mathbf{W}(\mathbf{G}^0, \mathfrak{h})$ is exactly the Weyl group $\mathbf{W}(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$ and $\mathfrak{h}$ are considered as complex Lie algebras. Thus the conclusion of Lemma 5.3 is trivially true.

From the results above it follows:

Theorem 5.6. *A quadric $\hat{Q}$ with a semisimple associated Levi-Tanaka algebra $\mathfrak{g}$ has the $(\tilde{S})$ property if and only if the simple factors of $\mathfrak{g}$ are all of the following real types:*

(1) $\mathrm{A}_\ell\,\mathrm{III/IV}$, $\Phi = \{\alpha_i\}$, *with* $1 \le i \le p$ *and* $i \ne (\ell+1)/2$ (*or* $q \le i \le \ell$ *and* $i \ne (\ell+1)/2$);
(2) $\mathrm{D}_\ell\,\mathrm{Ib/IIIb}$, $\Phi = \{\alpha_\ell\}$ (*or* $\Phi = \{\alpha_{\ell-1}\}$);
(3) $\mathrm{E\,II/III}$, $\Phi = \{\alpha_1\}$ (*or* $\Phi = \{\alpha_6\}$);

or of the following complex types:

(1′) $\mathrm{A}_\ell^{\mathbb{C}}$ *with* $\Phi = \{\alpha_j, \alpha'_{\ell+1-j}\}$;
(2′) $\mathrm{D}_\ell^{\mathbb{C}}$ *with ℓ even and* $\Phi = \{\alpha_1, \alpha'_{\ell-1}\}$ *or* $\Phi = \{\alpha_{\ell-1}, \alpha'_\ell\}$ (*or* $\Phi = \{\alpha_1, \alpha'_\ell\}$ *or* $\Phi = \{\alpha_{\ell-1}, \alpha'_1\}$, $\Phi = \{\alpha_\ell, \alpha'_{\ell-1}\}$ *or* $\Phi = \{\alpha_\ell, \alpha'_1\}$);
(3′) $\mathrm{D}_\ell^{\mathbb{C}}$ *with ℓ odd and* $\Phi = \{\alpha_\ell, \alpha'_{\ell-1}\}$ (*or* $\Phi = \{\alpha_{\ell-1}, \alpha'_\ell\}$);
(4′) $\mathrm{E}_6^{\mathbb{C}}$ *with* $\Phi = \{\alpha_1, \alpha'_6\}$ (*or* $\Phi = \{\alpha_6, \alpha'_1\}$).

Proof. The type listed are exactly those for which $\mathfrak{p}^{\mathrm{opp}}$ is conjugate to $\mathfrak{p}$. □

We will see later that for a semisimple $\mathfrak{g}$, the (S) property and the $(\tilde{S})$ property are equivalent.

6. The general case

We drop now the hypothesis that the Levi-Tanaka algebra associated to a quadric $\hat{Q}$ is semisimple.

Lemma 6.1. *If a quadric $\hat{Q}$ has property $(\tilde{S})$, then there exists a* $\mathrm{Ad}(\gamma)$*-invariant graded Levi factor $\mathfrak{s}$ of $\mathfrak{g}$, as described in* §4.

Proof. Taft [12] proves that if Γ is a finite group of automorphisms of a real Lie algebra $\mathfrak{g}$, and $\mathfrak{a} \subset \mathfrak{g}$ is a Γ-invariant semisimple subalgebra, then there exists a Γ-invariant Levi factor $\mathfrak{s}$ and a Γ-fixed element X in the nilradical of $\mathfrak{g}$ such that $\mathrm{Ad}(\exp X)(\mathfrak{a}) \subset \mathfrak{s}$. Actually his proof is valid for any Γ-invariant subalgebra $\mathfrak{a}$ contained in some (non necessarily invariant) Levi factor. It follows that if $\mathfrak{a}$ is a Γ-invariant subalgebra contained in some Levi factor, then there exists a Γ-invariant Levi factor $\mathfrak{s}$ containing $\mathfrak{a}$.

Let Γ be the group generated by $\mathrm{Ad}(\gamma)$, and let $\mathfrak{a} = \mathbb{C}\cdot E$. By [11] there exists a Levi factor containing $\mathfrak{a}$. It follows that there exists a Γ-invariant Levi factor $\mathfrak{s}$ containing $\mathfrak{a}$. It is graded, because it contains E, and it has a compatible complex structure on $\mathfrak{s}_{-1}$ again by [11]. □

We fix then a Levi-Malcev decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$ as in §4. Since the group $\mathbf{G}^0$ is semi-algebraic, we also have a corresponding Levi decomposition $\mathbf{G}^0 = \mathbf{SR}$ (note that $\mathbf{S} \cap \mathbf{R}$ is discrete, but not necessarily trivial).

Lemma 6.2. *If* $\mathfrak{p}^{\mathrm{opp}}$ *is* $\mathrm{Int}(\mathfrak{g})$*-conjugate to* $\mathfrak{p}$*, then* $\mathfrak{p}^{\mathrm{opp}} \cap \mathfrak{s}$ *is* $\mathrm{Int}(\mathfrak{s})$*-conjugate to* $\mathfrak{p} \cap \mathfrak{s}$*.*

Proof. Decompose an element $\gamma \in \mathrm{Int}(\mathfrak{g}) = \mathbf{G}^0$ such that $\mathrm{Ad}(\gamma)(\mathfrak{p}) = \mathfrak{p}^{\mathrm{opp}}$ as $\gamma = \gamma_{\mathbf{S}}\gamma_{\mathbf{R}}$. Then $\mathrm{Ad}(\gamma_{\mathbf{S}})(\mathfrak{p} \cap \mathfrak{s}) = \mathfrak{p}^{\mathrm{opp}} \cap \mathfrak{s}$. □

The simple ideals of the Levi factor $\mathfrak{s}$ belong to three families. Those of kind 2 are Levi-Tanaka algebras and, by Lemma 4.1, there is at least one of them. Those of kind 1 are of the complex type and correspond to compact hermitian symmetric spaces. We can ignore for the moment those of kind 0.

Theorem 6.3. *The quadric* $\hat{Q}$ *has property* $(\tilde{S})$ *if and only if* $E = E_{\mathfrak{s}}$ *and the simple ideals of kind* 2 *of a Levi factor are of the types described in Theorem* 5.6*, and the simple ideals of kind* 1 *of a Levi factor are of the following types:*

(1) $\mathrm{A}_\ell^{\mathbb{C}}$ *with* ℓ *odd and* $\Phi = \{\alpha_{(\ell+1)/2}\}$*;*
(2) $\mathrm{D}_\ell^{\mathbb{C}}$ *with* ℓ *even and* $\Phi = \{\alpha_1\}$ *or* $\Phi = \{\alpha_{\ell-1}\}$ *or* $\Phi = \{\alpha_\ell\}$*;*
(3) $\mathrm{D}_\ell^{\mathbb{C}}$ *with* ℓ *odd and* $\Phi = \{\alpha_1\}$*.* □

Proof. Indeed the same proof as in Theorem 5.6 applies to the Levi factor. The resulting element γ is still of finite order in $\mathbf{G}$, because $\mathbf{S}$ is a finite covering of $\mathrm{Int}(\mathfrak{s})$. □

7. Recovering an involution

So far we have proved only the existence of a finite order inner automorphism reversing the degree. Now we investigate the existence of an involutive automorphism with this property.

We keep the notation of the previous section. Moreover, let $\hat{\mathbf{S}}$ be the universal connected linear group with Lie algebra $\mathfrak{s}$, i.e. the set of real points of the simply connected group with Lie algebra $\mathfrak{s}^{\mathbb{C}}$. There is a natural projection $\pi\colon \hat{\mathbf{S}} \to \mathbf{S}$ which is a finite covering map.

We proceed in two steps. For simple Levi factors of kind 2, we look for elements $\gamma, \gamma' \in \hat{\mathbf{S}}$, with the properties that: (i) $\mathrm{Ad}_{\mathfrak{g}}(\gamma)(E) = \mathrm{Ad}_{\mathfrak{g}}(\gamma')(E) = -E$, (ii) $\gamma'^2 = e$, (iii) $\gamma^2 \in \mathbf{Z}(\hat{\mathbf{S}})$, and $\gamma^2|_{V^\lambda} = (-1)^{2\lambda(E)}$ for every irreducible representation V and weight λ. For simple Levi factors of kind 1, we provide a general construction for such an element γ. In many cases the image of γ^2 or γ'^2 in $\mathbf{S}$, and hence in $\mathbf{G}$, is the identity, and thus we obtain the (S) property.

We remark that in the following discussion the algebraic structure of the radical doen not play any role, and we only consider it as a $\mathfrak{s}$-module.

We introduce the following notation. If α is a root of $\mathfrak{s}$, then let $\mathfrak{s}(\alpha)$ be the (complex) Lie subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ containing $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{-\alpha}$, and $\mathbf{S}(\alpha)$ the corresponding analytic subgroup in $\hat{\mathbf{S}}$. Let $\tilde{s}_\alpha$ be the image of $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$ in $\mathbf{S}(\alpha)$. We have that $\mathrm{Ad}(\tilde{s}_\alpha) = s_\alpha$, $\tilde{s}_\alpha^4 = 1$, and $\tilde{s}_\alpha^2|_{V^\lambda} = (-1)^{(\alpha^\vee,\lambda)}$ for any representation V and weight λ.

7.1. **Simple ideals of kind 2.**

Case A_ℓ. In this case $\lambda(E) \in \mathbb{Z}$ for all weights λ. Denote by A_k the $k\times k$ matrix with entries equal to 1 on the antidiagonal, and 0 elsewhere. Let $\gamma \in \mathbf{S}$ be the block matrix:

$$\gamma = \begin{pmatrix} & & A_{[\frac{\ell}{2}]} \\ & B & \\ A_{[\frac{\ell}{2}]} & & \end{pmatrix}$$

with $B = (1), (-1), I_2, A_2$ depending on the class of ℓ modulo 4, in such a way that $\det\gamma = 1$. Then γ satisfies our hypotheses.

Case D_ℓ *with* $\ell = 2k+1$ *odd.* In this case $\lambda(E) \in \mathbb{Z}$ for all weights λ. Let $\tilde{w}_0 = \Pi_{i=1}^k \tilde{s}_{e_{2i-1}+e_{2i}} \tilde{s}_{e_{2i-1}-e_{2i}}$. Then, if $\{\omega_j\}$ are the fundamental weights,

$$\tilde{w}_0^2|_{V^{\omega_j}} = \begin{cases} \mathrm{Id} & \text{if } 1 \le j \le 2k-1, \\ (-1)^k \mathrm{Id} & \text{if } j = 2k, 2k+1. \end{cases}$$

If k is even, i.e. $\ell \equiv 1 \pmod 4$, then $\gamma = \gamma' = \tilde{w}_0$ satisfies $\gamma^2 = 1$.

If k is odd, i.e. $\ell \equiv 3 \pmod 4$, then we identify the subalgebra corresponding to $\{\pm\alpha_{\ell-2}, \pm\alpha_{\ell-1}, \pm\alpha_\ell\}$ to $\mathfrak{su}(1,3)$ or $\mathfrak{su}(2,2)$ or $\mathfrak{sl}(4,\mathbb{C})$ and let h be the image of $i\mathrm{Id}$ in the corresponding subgroup. Then $\tilde{w}_0$ and h commute, and $\gamma = \gamma' = \tilde{w}_0 h$ is the sought after element.

Case $\mathrm{D}_\ell\,\mathrm{Ib}$ *or* $\mathrm{D}_\ell^{\mathbb{C}}$ *with* $\ell = 2k$ *even,* $\Phi = \{\alpha_{\ell-1}\}$ *or* $\Phi = \{\alpha_{\ell-1}, \alpha'_\ell\}$. In this case $\omega_j(E) = j \in \mathbb{Z}$ for the fundamental weights $\omega_1, \ldots, \omega_{\ell-2}$, and $\omega_{\ell-1}(E) = \omega_\ell(E) = (\ell-1)/2$. Let $\tilde{w}_0 = \Pi_{i=1}^k \tilde{s}_{e_{2i-1}+e_{2i}} \tilde{s}_{e_{2i-1}-e_{2i}}$. Then, if $\{\omega_j\}$ are the fundamental weights,

$$\tilde{w}_0^2|_{V^{\omega_j}} = \begin{cases} \mathrm{Id} & \text{if } 1 \le j \le 2k-2, \\ (-1)^k \mathrm{Id} & \text{if } j = 2k-1, 2k. \end{cases}$$

If k is odd, i.e. $\ell \equiv 2 \pmod 4$, then $\gamma = \tilde{w}_0$ satisfies $\gamma^2|_{V^\lambda} = (-1)^{2\lambda(E)}$. Let $I \in \mathbf{Spin}(\ell-1,\ell+1)$ be an element covering $-\mathrm{Id} \in \mathbf{SO}(\ell-1,\ell+1)$. Then $\gamma' = (I \cdot \tilde{w}_0)$ satisfies $\gamma'^2 = \mathrm{Id}$

If k is even, i.e. $\ell \equiv 0 \pmod 4$, then $\gamma' = \tilde{w}_0$ satisfies $\gamma'^2 = \mathrm{Id}$. In general however it is not possible to find an element γ with the required properties.

Case $D^{\mathbb{C}}_\ell$ *with* $\ell = 2k$ *even,* $\Phi = \{\alpha_1, \alpha'_{\ell-1}\}$. In this case $\omega_i(E) \in \frac{1}{2}\mathbb{Z}$ for all fundamental weights ω_i, and $\omega_i(E) \in \mathbb{Z}$ exactly for $\omega_2, \omega_4, \dots, \omega_{2k-2}$ and for $\omega_{\ell-1}$ (resp. ω_ℓ) if $\ell \equiv 0 \pmod 4$ (resp. if $\ell \equiv 2 \pmod 4$).

As in the previous case, there exists an element γ' with $\gamma'^2 = e$ satisfying all conditions. In general however it is not possible to find an element γ with the required properties.

Case E_6. In this case $\lambda(E) \in \mathbb{Z}$ for all weights λ. Let

$$\gamma = \gamma' = \tilde{w}_0 = \tilde{s}_{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6}\tilde{s}_{\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6}\tilde{s}_{\alpha_4}\tilde{s}_{\alpha_3+\alpha_4+\alpha_5}.$$

Then $\gamma^2 = e$

Summarizing, we found an element γ' for all simple factors of kind 2, and an element γ for all simple factors of kind 2 excepts some those of kind D_ℓ with ℓ even.

7.2. **Simple ideals of kind** 1. First we consider the existence of a suitable element γ. The longest element w_0 of the Weyl group can be written as a product of reflections

$$w_0 = \Pi s_{\beta_i}$$

where $\{\beta_i\}$ is a maximal set of positive strongly orthogonal roots. Let $\{\alpha_j\} \subset \{\beta_i\}$ be the subset of roots of degree 1 (i.e. $\alpha_j(E) = 1$), and $w_1 = \Pi s_{\alpha_j}$, $\gamma = \Pi \tilde{s}_{\alpha_j}$.

Since $w_1(E) = -E$, we have

$$E = \sum_j \frac{\alpha_j(E)\alpha_j}{(\alpha_j, \alpha_j)} = \frac{1}{2}\sum_j \alpha_j(E)\alpha_j^\vee = \frac{1}{2}\sum_j \alpha_j^\vee.$$

Then

$$\gamma|_{V^\lambda} = \Pi_j(-1)^{(\alpha_j^\vee,\lambda)} = (-1)^{(\sum_j \alpha_j^\vee,\lambda)} = (-1)^{2\lambda(E)}.$$

We turn now to the problem of the existence of γ' with $\gamma'^2 = 1$. For simple ideals of type $D^{\mathbb{C}}_\ell$ or $E^{\mathbb{C}}_6$ the element γ' found in the previous subsection is a representative of the longest element of the Weyl group, thus satisfies all requirements. For simple ideals of type $A^{\mathbb{C}}_\ell$ with $\ell \equiv 3 \pmod 4$, the matrix with entries equal to 1 on the antidiagonal and 0 elsewhere provides the element γ'. For simple ideals of type $A^{\mathbb{C}}_\ell$ with $\ell \equiv 1 \pmod 4$ there is no such element γ'.

Theorem 7.1. *Let* $\hat{Q}$ *be a quadric with the* $(\tilde{S})$ *property, and* $\mathfrak{g}$ *the associated Levi-Tanaka algebra, with Levi-Malcev decomposition* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$. *If any of the following conditions is satisfied, then* $\hat{Q}$ *has the* (S) *property:*

1. $\mathfrak{g}$ *is semisimple;*
2. $\mathfrak{s}$ *does not contain any simple factor of kind* 1 *and type* $A^{\mathbb{C}}_\ell$ *with* $\ell \equiv 1 \pmod 4$,
3. $\mathfrak{s}$ *does not contain any simple factor of kind* 2 *and type* D_ℓ *with* ℓ *even.*

Proof. In case (1) all the ideals of $\mathfrak{s}$ are of kind 2, so case (1) is a subcase of case (2).

For each simple ideal $\mathfrak{s}_i$ of $\mathfrak{s}$, let γ_i, γ'_i be the images in $\mathbf{S}_i$ of the elements described in the previous sections, if defined. For Levi factors of kind 0 we let $\gamma_i = \gamma'_i = e$.

In case (2) the elements γ_i are defined for every simple factor $\mathfrak{s}_i$, and we let $\gamma = \Pi_i \gamma_i$. In case (3) the elements γ'_i are defined for every simple factor $\mathfrak{s}_i$, and we let $\gamma = \Pi_i \gamma'_i$. In both cases the element $\gamma \in \mathbf{G}$ has order 2. □

Since compact quadrics have a semisimple group of automorphisms, we have the following.

Corollary 7.2. *Every compact CR quadric has propoerty* (S). □

Remark 7.3. If a quadric has property $(\tilde{S})$ the above construction shows that it is anyway possible find an appropriate automorphism γ with order 2 or 4.

Remark 7.4. As the next example shows, the conditions in Theorem 7.1 are not necessary. In fact we could not find any example of quadrics with the $(\tilde{S})$ property but without the (S) property.

Example 7.5. Let $\mathfrak{s} = \mathfrak{o}(8, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, with the grading and the CR structure defined respectively by:

$$E = \operatorname{diag}(1, 1, 1, 0, 0, -1, -1, -1) \oplus \operatorname{diag}(1/2, -1/2),$$
$$J = \operatorname{diag}(0, 0, 0, \mathrm{i}, -\mathrm{i}, 0, 0, 0) \oplus \operatorname{diag}(\mathrm{i}/2, -\mathrm{i}/2).$$

Let $\mathbb{C}^8$ and $\mathbb{C}^2$ denote the standard representations of $\mathfrak{o}(8, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})$, respectively, and let $V = \mathbb{C}^8 \otimes \mathbb{C}^2$. The same elements E and J define a grading and CR structure on the semidirect product $\mathfrak{s} \oplus V$. We finally define $\mathfrak{g} = \mathfrak{s} \oplus V \oplus \mathbb{C}T$, where T is an element commuting with $\mathfrak{s}$ and such that $\operatorname{ad}(T)|_V = \operatorname{Id}$. Then $\mathfrak{g}$ is a Levi-Tanaka algebra associated to a quadric with the $(\tilde{S})$ property. It has the (S) property too, with the element $\gamma = \gamma'_{\mathfrak{o}(8,\mathbb{C})} \gamma_{\mathfrak{sl}(2,\mathbb{C})} \exp(\mathrm{i}\pi T/2)$, but there is no such element γ in $\mathbf{S}$.

8. An example

The CR dimension of a quadric $\hat{Q}$ is $n = \dim_{\mathbb{R}} \mathfrak{g}_{-1}/2$, while the CR-codimension is $k = \dim_{\mathbb{R}} \mathfrak{g}_{-2}$, and hence $\dim_{\mathbb{R}} \mathfrak{g}_{-} = 2n + k$. For quadrics with the (S) or $(\tilde{S})$ property, the dimension of $\mathfrak{g}_{+} = \operatorname{Ad}(\gamma)(\mathfrak{g}_{-})$ is $2n + k$. It was an open question whether also in the general case the dimension of $\mathfrak{g}_{+}$ can be estimated by $2n + k$ (see [6, p.445]). The following example gives a negative answer to this question.

Example 8.1. Consider $\mathfrak{s} = \mathfrak{sl}(3, \mathbb{C}) = \bigoplus_{i=-2}^{2} \mathfrak{s}_i$ endowed with the unique Levi-Tanaka structure, given by the elements

$$E^{\mathfrak{s}} = \operatorname{diag}(1, 0, -1), \qquad J^{\mathfrak{s}} = \operatorname{diag}(-\mathrm{i}/3, 2\mathrm{i}/3, -\mathrm{i}/3).$$

Let $V = \mathbb{C}^3$ be the space of the standard representation ρ of $\mathfrak{s}$ and U^1, U^2 two copies the adjoint representation of $\mathfrak{s}$. We assume on V the grading $V = V_{-2} + V_{-1} + V_0$ given by the eigenspace decomposition of $(\rho(E^{\mathfrak{s}}) - \mathrm{Id})$ and on V_{-1} the complex structure given by multiplication by the imaginary unit $J = \mathrm{iId}$.

On $U^k, k = 1, 2$, we put the grading $U^k = \oplus_{i=-2}^{2} U_i^k$ induced by $\mathrm{Ad}(E^{\mathfrak{s}})$ and the complex structure induced by $\mathrm{Ad}(J^{\mathfrak{s}})$.

On $\mathfrak{h} = \mathfrak{s} \oplus V \oplus U^1 \oplus U^2$ we have a natural Lie algebra structure, with $V \oplus U^1 \oplus U^2$ an abelian ideal and $\mathfrak{s}$ acting through the standard or adjoint representation. Then $\mathfrak{h}$ is a graded CR Lie algebra, and it is fundamental, nondegenerate and transitive.

Let W^1, W^2 be two copies to the dual space V^* of V. The algebra $\mathfrak{s}$ acts on them via the contragradient representation $-{}^t\rho$. We assume on $W^k, k = 1, 2$ a grading $W^k = W_0^k + W_1^k + W_2^k$ given by the eigenspace decomposition of $(-{}^t\rho(E^{\mathfrak{s}}) + \mathrm{Id})$. We define a product of elements of V and W^k

$$[v, w] := v \otimes w$$

with values on $V \otimes W^k$, which we identify with $U^k \oplus \mathbb{C} \simeq \mathfrak{gl}(n, \mathbb{C})$.

Assuming $W^1 + W^2 + U^1 + U^2 + \mathbb{C}^2$ abelian, we obtain a graded Lie algebra $\mathfrak{a} = \mathfrak{s} + V + W^1 + W^2 + U^1 + U^2 + \mathbb{C}^2$ which is nondegenerate and fundamental. It is also pseudocomplex and transitive (see [8]). Its maximal pseudocomplex prolongation $\mathfrak{g} = \oplus_{-2}^{2} \mathfrak{g}_i$ is finite dimensional with $\dim \mathfrak{g}_1 \geq \dim \mathfrak{a}_1 = 8 > 7 = \dim \mathfrak{g}_{-1}$ and $\dim \mathfrak{g}_2 \geq \dim \mathfrak{a}_1 = 5 > 4 = \dim \mathfrak{g}_{-1}$.

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A. Altomani: Research Unit in Mathematics, University of Luxembourg, 6, rue Coudenhove-Kalergi, L-1359 Luxembourg
E-mail address: andrea.altomani@uni.lu

C. Medori: Dipartimento di Matematica, Università di Parma, Parco Area delle Scienze 53/A, 43124 Parma (Italy)
E-mail address: costantino.medori@unipr.it