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ON HOMOGENEOUS *CR* MANIFOLDS AND THEIR *CR* ALGEBRAS

ANDREA ALTOMANI

Dipartimento di Matematica
II Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 00133 Roma, Italy
altomani@mat.uniroma2.it

COSTANTINO MEDORI

Dipartimento di Matematica, Università di Parma
Parco Area delle Scienze 53/A, 43100 Parma, Italy
costantino.medori@unipr.it

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In this paper we show some results on homogeneous *CR* manifolds, proved by introducing their associated *CR* algebras. In particular, we give different notions of nondegeneracy (generalizing the usual notion for the Levi form) which correspond to geometrical properties for the corresponding manifolds. We also give distinguished equivariant *CR* fibrations for homogeneous *CR* manifolds. In the second part of the paper we apply these results to minimal orbits for the action of a real form of a semisimple Lie group $\hat{\mathbf{G}}$ on a flag manifold $\hat{\mathbf{G}}/\mathbf{Q}$.

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0. Introduction

The study of the automorphisms group of *CR* manifolds was pioneered in 1932 by É. Cartan, who investigated the holomorphic invariants of real hypersufaces in $\mathbb{C}^2$. It was carried on for real hypersurfaces in $\mathbb{C}^n$ by Morimoto and Nagano (see [15]) and continued by Chern and Moser, and Tanaka in the 1970s (see in particular [9, 20]).

In this context homogeneous *CR* manifolds are natural objects to consider. Orbits for the action of a real Lie group of biholomorphisms of a complex manifold provide large classes of examples. In particular, the orbits of a real form $\mathbf{G}$ of a complex semisimple Lie group $\hat{\mathbf{G}}$ in a flag manifold $\hat{\mathbf{G}}/\mathbf{Q}$ have been investigated by Wolf [21] and later by several other authors, also in connection with representation theory (see for example [8, 10]). Only one orbit of $\mathbf{G}$ in $\hat{\mathbf{G}}/\mathbf{Q}$ is compact. It has minimal dimension and is called *minimal orbit*.

In [19] Tanaka associated to any (regular) *CR* manifold M with nondegenerate Levi form a graded Lie algebra $\mathfrak{g}$ and, by a generalization of the theory of prolongation (in the sense of Sternberg), proved that the group of *CR* automorphisms of M is a Lie group whose dimension is bounded by $\dim \mathfrak{g}$ (see also [1]). Maximally homogeneous *CR* manifolds were studied by Nacinovich and the second author (see [11–13]). They are compact if and only if they are minimal orbits, but not all minimal orbits are maximally homogeneous. The notion of *CR* algebra and more general conditions of nondegeneracy were introduced in [14], allowing an in-depth study of the *CR* structure of minimal orbits (see [4]).

On the other hand, the study of compact homogeneous *CR* manifolds of hypersurface type was carried on by Azad *et al.* [6] and, more recently, by Alekseevsky and Spiro (see [18, 2, 3]), who completed the classification of all the simply connected ones.

In this paper we present some recent results on homogeneous *CR* manifolds obtained by Nacinovich and the authors by introducing *CR* algebras (see [14, 4]).

In the first part (Secs. 1–4) we deal with general homogeneous *CR* manifolds. After setting basic definitions and notation, we introduce the *CR algebra* associated to a homogeneous *CR* manifold M (Sec. 2). This is a pair $(\mathfrak{g}, \mathfrak{q})$ consisting of a real Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{q}$ of its complexification: it gives an infinitesimal description of the *CR* structure of M.

Next (Sec. 3) we recall the definition of a *CR* manifold of finite type and different nondegeneracy conditions, to introduce analogous definitions for *CR* algebras and state an equivalence result between the conditions on *CR* manifolds and those on *CR* algebras.

In Sec. 4 we introduce the important notion of equivariant *CR* fibrations for manifolds and algebras. In particular, we describe the fundamental and the weakly nondegenerate reductions of *CR* algebras. For (locally) homogeneous *CR* manifolds they correspond to local equivariant *CR* fibrations that generalize the classical Levi foliation of Levi-flat *CR* manifolds.

The second part of the paper (Secs. 5–9) consists of applications of these techniques to *minimal orbits* of a real semisimple Lie group $\mathbf{G}$ in a flag manifold $\hat{\mathbf{G}}/\mathbf{Q}$ of its complexification.

The corresponding *CR* algebras (*parabolic minimal*) can be effectively described in terms of the root system of $\hat{\mathfrak{g}}$ with respect to a suitably chosen Cartan subalgebra, and are classified by their *cross-marked Satake diagrams* (Secs. 6 and 7).

In the last two sections (Secs. 8 and 9) we describe equivariant *CR* fibrations of minimal orbits with the aid of cross-marked Satake diagram and show that the fiber is still a minimal orbit, whose Satake diagram can be computed from those of the base and the total space. Then we focus on the fundamental reduction. The authors wish to thank G. Fels for pointing out a mistake in the preliminary version of this paper.

1. Preliminaries on *CR* Manifolds

An (abstract) almost *CR* manifold of type (n, k) is a triple (M, HM, J), consisting of a paracompact smooth manifold M of real dimension $(2n+k)$, a smooth subbundle HM of TM of even rank $2n$ (its *holomorphic tangent space*) and a smooth partial complex structure $J : HM \to HM$, $J^2 = -\mathrm{Id}$, on the fibers of HM. The integer $n \geq 0$ is the *CR dimension* and k is the *CR codimension* of (M, HM, J).

Let $T^{1,0}M$ and $T^{0,1}M$ be the complex subbundles of the complexification $\mathbb{C} \otimes HM$ of HM, which correspond to the i- and $(-i)$-eigenspaces of J:

$$T^{1,0}M = \{X - i\,J\,X \,|\, X \in HM\}, \quad T^{0,1}M = \{X + i\,JX \,|\, X \in HM\}\,.$$

We say that (M, HM, J) is a *CR manifold* if the formal integrability condition:

$$[\mathcal{C}^\infty(M, T^{0,1}M), \mathcal{C}^\infty(M, T^{0,1}M)] \subset \mathcal{C}^\infty(M, T^{0,1}M) \tag{1.1}$$

holds (we get an equivalent condition by substituting $T^{1,0}$ for $T^{0,1}$ in (1.1)). When $k = 0$, we have $HM = TM$ and, via the Newlander–Nirenberg theorem, we recover the definition of a complex manifold. A smooth real manifold of real dimension k can always be considered as a *totally real CR* manifold, i.e. a *CR* manifold of *CR* dimension 0 and *CR* codimension k.

Let (M_1, HM_1, J_1), (M_2, HM_2, J_2) be two abstract smooth *CR* manifolds. A smooth map $f : M_1 \to M_2$, with differential $f_* : TM_1 \to TM_2$, is a *CR map* if $f_*(HM_1) \subset HM_2$, and $f_*(J_1 v) = J_2 f_*(v)$ for every $v \in HM_1$. We say that f is a *CR diffeomorphism* if $f : M_1 \to M_2$ is a smooth diffeomorphism and both f and f^{-1} are *CR* maps.

A *CR function* is a *CR* map $f : M \to \mathbb{C}$ of a *CR* manifold (M, HM, J) to $\mathbb{C}$, endowed with the standard complex structure.

A *CR embedding* ϕ of an abstract *CR* manifold (M, HM, J) into a complex manifold $\mathbf{X}$, with complex structure $J_{\mathbf{X}}$, is a *CR* map which is a smooth embedding and satisfies $\phi_*(H_pM) = \phi_*(T_pM) \cap J_{\mathbf{X}}(\phi_*(T_pM))$ for every $p \in M$.

If $\phi : M \to \mathbf{X}$ is a smooth embedding of a paracompact smooth manifold M into a complex manifold $\mathbf{X}$, for each point $p \in M$ we can define H_pM to be the set of tangent vectors $v \in T_pM$ such that $J_{\mathbf{X}}\phi_*(v) \in \phi_*(T_pM)$. For $v \in H_pM$, let $J_M v$ be the unique tangent vector $w \in H_pM$ satisfying $\phi_*(w) = J_{\mathbf{X}}\phi_*(v)$. If the dimension of H_pM is constant for $p \in M$, then $HM = \bigcup_{p \in M} H_pM$ and J_M are smooth and define the unique *CR* structure on M for which (M, HM, J_M) is a *CR* manifold and $\phi : M \to \mathbf{X}$ is a *CR* embedding. In particular, this is the case when M is an orbit for the action of a Lie group of biholomorphisms of $\mathbf{X}$.

In the next sections, to shorten notation, we shall write simply M, or $M^{n,k}$, for a *CR* manifold (M, HM, J) of type (n, k), when the *CR* structure will be clear from the context. Moreover, we shall denote by $\hat{V}$ and $\hat{\phi}$ the complexification of a vector space V and a linear map ϕ, respectively.

2. The *CR* Algebra Associated to a Homogeneous *CR* Manifold

Let (M, HM, J) be a *CR* manifold, which is homogeneous for the action of a real Lie group $\mathbf{G}$ of *CR* transformations, so that $M = \mathbf{G}/\mathbf{G}_+$ for a closed isotropy subgroup $\mathbf{G}_+$ of $\mathbf{G}$. Set $eG_+ = o \in M$.

To M we can associate a *CR algebra* $(\mathfrak{g}, \mathfrak{q})$. This is a pair consisting of the real Lie algebra $\mathfrak{g}$ of the group $\mathbf{G}$ and of a complex subalgebra $\mathfrak{q}$ of its complexification $\hat{\mathfrak{g}}$. This subalgebra is the inverse image

$$\mathfrak{q} = \hat{\pi}_*^{-1}(T_o^{0,1}M) \tag{2.1}$$

by the complexification $\hat{\pi}_*$ of the differential $\pi_* : \mathfrak{g} \simeq T_e\mathbf{G} \to T_oM$ of the group action at e. The fact that $\mathfrak{q}$ is a complex Lie subalgebra of $\hat{\mathfrak{g}}$ is a consequence of the formal integrability condition (1.1) for *CR* manifolds.

The real Lie subalgebra $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ is called *isotropy* and the subspace $\mathcal{H}_+ = (\mathfrak{q} + \bar{\mathfrak{q}}) \cap \mathfrak{g}$ is called *holomorphic tangent space* (where the conjugation is taken with respect to the real form $\mathfrak{g}$ of $\hat{\mathfrak{g}}$).

Vice versa, let $(\mathfrak{g}, \mathfrak{q})$ be a *CR* algebra. Let $\mathbf{G}$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Assume that the analytic subgroup $\mathbf{G}_+$ of $\mathbf{G}$ corresponding to the Lie subalgebra $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ is closed in $\mathbf{G}$. Then the homogeneous space $M = \mathbf{G}/\mathbf{G}_+$ is a smooth paracompact manifold and has a unique *CR* structure such that :

- $T_o^{0,1}M = \hat{\pi}_*(\mathfrak{q})$;
- $\mathbf{G}$ acts on M by *CR* diffeomorphisms.

We denote the *CR* manifold $\mathbf{G}/\mathbf{G}_+$ by $\tilde{M}(\mathfrak{g}, \mathfrak{q})$.

Let $(\mathfrak{g}, \mathfrak{q})$ be a *CR* algebra of type (n, k). We call $(\mathfrak{g}, \mathfrak{q})$:

- *totally real* if $\mathfrak{q} = \hat{\mathfrak{g}}_+ = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_+$, i.e. if $\mathfrak{g}_+ = \mathcal{H}_+$ or, equivalently, the *CR* dimension n is 0;
- *totally complex* if $\mathfrak{q} + \overline{\mathfrak{q}} = \hat{\mathfrak{g}}$, i.e. if $\mathcal{H}_+ = \mathfrak{g}$, or, equivalently, the *CR* codimension k is 0;
- *transitive*, or *effective*, if $\mathfrak{g}_+$ does not contain any nonzero ideal of $\mathfrak{g}$.

If $(\mathfrak{g}, \mathfrak{q})$ is the CR algebra associated to a $\mathbf{G}$-homogeneous CR manifold M, these notions express geometric properties of M (see [14]). In particular, effectiveness is equivalent to almost effectiveness of the $\mathbf{G}$ action (i.e. discreteness of the ineffective subgroup).

We can always reduce to the case of an almost effective action of $\mathbf{G}$: at the level of *CR* algebras, this corresponds to substituting to $(\mathfrak{g}, \mathfrak{q})$ its *effective quotient*, which is the *CR* algebra $(\mathfrak{g}/\mathfrak{a}, \mathfrak{q}/\hat{\mathfrak{a}})$, where $\mathfrak{a}$ is the maximal ideal of $\mathfrak{g}$ that is contained in $\mathfrak{g}_+$, and $\hat{\mathfrak{a}}$ its complexification in $\hat{\mathfrak{g}}$ (see [14, Sec. 4]).

3. Finiteness and Nondegeneracy Conditions

We recall (see [14, Sec. 13]) that a CR manifold M is:

- of *finite kind* (or *finite type*) at $p \in M$ if the higher order commutators of $C^\infty(M, HM)$, evaluated at p, span T_pM;
- *holomorphically nondegenerate* at $p \in M$ if there is no germ of nonzero holomorphic vector field X at p that is tangent to M at all points of a neighborhood of p (see [7, Sec. 11.3]);
- *weakly nondegenerate* at $p \in M$ if for each $\bar{Z} \in C^\infty(M, T^{0,1}M)$, with $\bar{Z}_{|p} \neq 0$, there exist $m \in \mathbb{N}$ and $Z_1, \ldots, Z_m \in C^\infty(M, T^{1,0}M)$ such that:
$$[Z_1, \ldots, Z_m, \bar{Z}]_{|p} = [Z_1, [Z_2, \ldots, [Z_m, \bar{Z}] \ldots]]_{|p} \notin T_p^{1,0}M + T_p^{0,1}M\,;$$
- *strictly nondegenerate* (or *Levi nondegenerate*) at $p \in M$ if for each $\bar{Z} \in C^\infty(M, T^{0,1}M)$, with $\bar{Z}_{|p} \neq 0$, there exists $Z' \in C^\infty(M, T^{1,0}M)$ such that:
$$[Z', \bar{Z}]_{|p} \notin T_p^{1,0}M + T_p^{0,1}M\,.$$

Holomorphic nondegeneracy is in fact equivalent to weak nondegeneracy for homogeneous CR manifolds (see [22, 23]).

We now give analogous conditions for a *CR* algebra. Let $(\mathfrak{g}, \mathfrak{q})$ be a *CR* algebra of type (n, k). We say that $(\mathfrak{g}, \mathfrak{q})$ is:

- *fundamental* if $\mathcal{H}_+$ generates $\mathfrak{g}$ as a Lie algebra; or, equivalently, $\mathfrak{q} + \overline{\mathfrak{q}}$ generates $\hat{\mathfrak{g}}$ as a complex Lie algebra;
- *ideal nondegenerate*, if there is no ideal $\mathfrak{a}$ of $\mathfrak{g}$ with $\mathfrak{a} \subset \mathcal{H}_+$ and $\mathfrak{a} \not\subset \mathfrak{g}_+$;
- *weakly nondegenerate*, if there is no complex subalgebra $\mathfrak{q}' \neq \mathfrak{q}$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \overline{\mathfrak{q}}$;
- *strictly nondegenerate*, if $\{X \in \mathcal{H}_+ \mid [X, \mathcal{H}_+] \subset \mathcal{H}_+\} = \mathfrak{g}_+$.

Clearly we have the implications:

strictly nondegenerate $\Rightarrow$ *weakly nondegenerate* $\Rightarrow$ *ideal nondegenerate.*

Examples given in [14, Sec. 2] show that these conditions are not equivalent.

For effective *CR* algebras $(\mathfrak{g}, \mathfrak{q})$, ideal nondegenerate reduces to the statement:

there are no nontrivial ideals of $\mathfrak{g}$ *contained in* $\mathcal{H}_+$

and this condition is actually equivalent to *effective and ideal nondegenerate.*

The main motivation for considering *ideal nondegenerate CR* algebras is the following (see [14, Sec. 9]):

Theorem 3.1. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be an ideal nondegenerate effective fundamental CR algebra. Then* $\mathfrak{g}$ *is finite-dimensional.*

We have the following (see [14, Sec. 13]):

Theorem 3.2. *Let* (M, HM, J) *be a CR manifold of type* (n, k), *homogeneous for the action of a Lie group* $\mathbf{G}$, *and* $(\mathfrak{g}, \mathfrak{q})$ *be the corresponding CR algebra.*

Then (M, HM, J) *is of finite kind* (*resp. weakly, strictly nondegenerate*) *at all* $p \in M$ *if and only if the CR algebra* $(\mathfrak{g}, \mathfrak{q})$ *is fundamental* (*resp. weakly, strictly nondegenerate*). *Ideal nondegeneracy has not an equivalent notion for CR manifolds. In fact an ideal degenerate CR algebra and an ideal nondegenerate one may correspond to the same CR manifold* (*see* [22]).

4. Equivariant Fibrations and Reductions

A *morphism of CR algebras* $\phi : (\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}', \mathfrak{q}')$ is a homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ of real Lie algebras whose complexification $\hat{\phi}$ satisfies $\hat{\phi}(\mathfrak{q}) \subset \mathfrak{q}'$; it is a *CR submersion* if $\phi(\mathfrak{g}) + \mathfrak{g}'_+ = \mathfrak{g}'$ and $\hat{\phi}(\mathfrak{q}) + \mathfrak{q}' \cap \bar{\mathfrak{q}}' = \mathfrak{q}'$.

A *CR* map $f : M_1 \to M_2$ is a *CR fibration* if it is a smooth submersion, i.e. $f_*(TM_1) = TM_2$, and a *CR submersion*, i.e. $f_*(HM_1) = HM_2$. In particular this is the case if $\mathfrak{q}_1 \subset \mathfrak{q}_2$, $f : M_1 = \tilde{M}(\mathfrak{g}, \mathfrak{q}_1) \to \tilde{M}(\mathfrak{g}, \mathfrak{q}_2) = M_2$ is the natural projection and either $\mathfrak{q}_2 \subset \mathfrak{q}_1 + \bar{\mathfrak{q}}_1$ or $\mathfrak{q}_2 = \bar{\mathfrak{q}}_2$.

In this section we discuss morphisms of *CR* algebras of the special form

$$\Phi_{\mathfrak{q}}^{\mathfrak{q}'} : (\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \mathfrak{q}')\,, \quad \text{for } \mathfrak{q} \subset \mathfrak{q}'. \tag{4.1}$$

They have been called in [14] $\mathfrak{g}$-*equivariant fibrations* and describe at the level of *CR* algebras the corresponding $\mathbf{G}$-equivariant smooth *CR* fibrations $\tilde{M}(\mathfrak{g}, \mathfrak{q}) \to \tilde{M}(\mathfrak{g}, \mathfrak{q}')$.

The *CR* algebra $(\mathfrak{g}'', \mathfrak{q}'') = (\mathfrak{g}'_+, \mathfrak{q} \cap \bar{\mathfrak{q}}')$ is called the *fiber* of the $\mathfrak{g}$-equivariant fibration (4.1). It completely determines $\mathfrak{q}' \supset \mathfrak{q}$ by $\mathfrak{q}' = \mathfrak{q} + \hat{\mathfrak{g}}''$.

The $\mathfrak{g}$-equivariant fibration $\Phi_{\mathfrak{q}}^{\mathfrak{q}'} : (\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \mathfrak{q}')$ will also be written as:

$$(\mathfrak{g}, \mathfrak{q}) \xrightarrow[(\mathfrak{g}'', \mathfrak{q}'')]{} (\mathfrak{g}, \mathfrak{q}')\,, \tag{4.2}$$

with $(\mathfrak{g}, \mathfrak{q})$ the *total space*, $(\mathfrak{g}, \mathfrak{q}')$ *the base*, and $(\mathfrak{g}'', \mathfrak{q}'')$ *the fiber* of the fibration (4.2).

Now we describe two important examples of equivariant *CR* fibrations (see [14, Sec. 5]).

4.1. *Reduction to fundamental CR algebras*

Proposition 4.1. *Every CR algebra* $(\mathfrak{g}, \mathfrak{q})$ *admits a unique* $\mathfrak{g}$-*equivariant CR fibration* (4.2) *with a totally real base and a fundamental fiber*, (*see* [14, *Sec.* 5]).

We shall refer to the canonical $\mathfrak{g}$-equivariant fibration of Proposition 4.1 as the *fundamental reduction* of the *CR* algebra $(\mathfrak{g}, \mathfrak{q})$.

Let $\hat{\mathfrak{f}}$ be the subalgebra of $\hat{\mathfrak{g}}$ generated by $\mathfrak{q} + \bar{\mathfrak{q}}$. Then the base of the reduction is $(\mathfrak{g}, \hat{\mathfrak{f}})$ and the fiber is $(\mathfrak{g}'', \mathfrak{q}'') = (\mathfrak{g} \cap \hat{\mathfrak{f}}, \mathfrak{q})$.

Note that if $\Phi : (\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}', \mathfrak{q}')$ is a *CR* submersion and $(\mathfrak{g}, \mathfrak{q})$ is fundamental, then also $(\mathfrak{g}', \mathfrak{q}')$ is fundamental.

For general *CR* manifolds we have, by Frobenius Theorem:

Proposition 4.2. *Let* (M, HM, J) *be a CR manifold. Let* $B \subset TM$ *be the subbundle generated by* HM *and its commutators. If the rank of* B *is constant in a small* *neighborhood U of a point $p \in M$, then B generates a smooth local foliation $U = \bigcup U_\alpha$ such that each leaf U_α is a CR manifold of finite type and its CR dimension is the same as the CR dimension of M.*

Furthermore, if M is locally $\mathbf{G}$-homogeneous for a Lie group $\mathbf{G}$ of CR transformations, then every U_α is locally homogeneous and the associated CR algebra is $(\mathfrak{g}'', \mathfrak{q}'')$.

4.2. *Reduction to weakly nondegenerate CR algebras*

Proposition 4.3. *Every fundamental CR algebra $(\mathfrak{g}, \mathfrak{q})$ admits a unique $\mathfrak{g}$-equivariant CR fibration (4.2) with a weakly nondegenerate fundamental base and a totally complex fiber (see [14, Sec. 5]).*

We shall refer to the canonical $\mathfrak{g}$-equivariant fibration of Proposition 4.3 as the *weakly nondegenerate reduction* of the fundamental *CR* algebra $(\mathfrak{g}, \mathfrak{q})$.

When $(\mathfrak{g}, \mathfrak{q})$ is weakly degenerate, it was proved in [14] that there is a *CR*-fibration $\tilde{M}(\mathfrak{g}, \mathfrak{q}) \to M'$ of the corresponding homogeneous *CR* manifold $\tilde{M}(\mathfrak{g}, \mathfrak{q})$ on a *CR* manifold M' with the same *CR* codimension, having a nontrivial complex fiber. For homogeneous simply connected *CR* manifolds, the condition of weak degeneracy is in fact necessary and sufficient for the existence of *CR* fibrations with nontrivial complex fibers. Indeed, for general *CR* manifolds, the existence of a *CR* fibration with nontrivial complex fibers implies weak degeneracy, as we have:

Proposition 4.4. *Let M and M' be CR manifolds. We assume that M' is locally embeddable and that there exists a CR fibration $M \xrightarrow{\pi} M'$ with totally complex fibers of positive dimension. Then M is weakly degenerate.*

Proof. Let f be any smooth *CR* function defined on a neighborhood U' of $p' \in M'$. Then $\pi^* f$ is a *CR* function in $U = \pi^{-1}(U')$, that is constant along the fibers of π. Then, if $L \in \mathcal{C}^\infty(M, T^{1,0}M)$ is tangent to the fibers of π in U, we obtain that $[\bar{Z}_1, \dots, \bar{Z}_m, L](\pi^* f) = 0$ for every choice of $\bar{Z}_1, \dots, \bar{Z}_m \in \mathcal{C}^\infty(M, T^{0,1}M)$. Assume by contradiction that M is weakly nondegenerate at some p with $\pi_{|p} = p'$. Then for some choice of $\bar{Z}_1, \dots, \bar{Z}_m \in \mathcal{C}^\infty(M, T^{0,1}M)$ we would have $v_p = [\bar{Z}_1, \dots, \bar{Z}_m, L] \notin T^{1,0}_p M \oplus T^{0,1}_p M$. Since the fibers of π are totally complex, $\pi_*(v_p) \neq 0$. By the assumption that M' is locally embeddable at p, the real parts of the (locally defined) *CR* functions give local coordinates in M' and therefore there is a *CR* function f defined on a neighborhood U' of p' with $v_p(\pi^* f) = \pi_*(v_p)(f) \neq 0$. This gives a contradiction, proving our statement. □

4.3. *The reduction diagram*

Let $(\mathfrak{g}, \mathfrak{q})$ be a *CR* algebra. Let us denote by $\mathfrak{f}$ the Lie subalgebra of $\mathfrak{g}$ generated by $\mathcal{H}_+(\mathfrak{g}, \mathfrak{q})$ and by $\mathfrak{q}'$ the largest subalgebra of $\hat{\mathfrak{g}}$ such that $\mathfrak{q} \subset \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$. Next we set

$\mathfrak{g}'' = \mathfrak{f} \cap \mathfrak{q}'$ and $\mathfrak{q}'' = \mathfrak{q} \cap \hat{\mathfrak{g}}''$. Then the canonical fibrations discussed in this section can be summarized in the diagram:

$$\begin{array}{ccc} (\mathfrak{g}, \mathfrak{q}) & & \\ \big\downarrow (\mathfrak{f}, \mathfrak{q}) & \xrightarrow[(\mathfrak{g}'', \mathfrak{q}'')]{} & (\mathfrak{f}, \mathfrak{q}'), \\ (\mathfrak{g}, \hat{\mathfrak{f}}) & & \end{array} \tag{4.3}$$

where $(\mathfrak{g}, \hat{\mathfrak{f}})$ is totally real, $(\mathfrak{f}, \mathfrak{q})$ is fundamental, $(\mathfrak{g}'', \mathfrak{q}'')$ is totally complex, $(\mathfrak{f}, \mathfrak{q}')$ is fundamental and weakly nondegenerate.

5. Minimal Orbits in Flag Manifolds

A *complex flag manifold* is a coset space $\mathbf{X} = \hat{\mathbf{G}}/\mathbf{Q}$, where $\hat{\mathbf{G}}$ is a connected complex semisimple Lie group and $\mathbf{Q}$ is parabolic in $\hat{\mathbf{G}}$. The manifold $\mathbf{X}$ is a closed complex projective variety which only depends on the Lie algebras $\hat{\mathfrak{g}}$ of $\hat{\mathbf{G}}$ and $\mathfrak{q}$ of $\mathbf{Q}$: this is a consequence of the fact that the center of a connected and simply connected complex Lie group is contained in each of its parabolic subgroups.

A *real form* of $\hat{\mathbf{G}}$ is a real subgroup $\mathbf{G}$ of $\hat{\mathbf{G}}$ whose Lie algebra $\mathfrak{g}$ is a real form of $\hat{\mathfrak{g}}$ (i.e. $\hat{\mathfrak{g}} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$). The real form $\mathfrak{g}$ is the set of fixed points of an anti-involution σ, called *conjugation* in $\hat{\mathfrak{g}}$: $\mathfrak{g} = \mathrm{Fix}_{\hat{\mathfrak{g}}}(\sigma) = \{X \in \hat{\mathfrak{g}} \mid \sigma(X) = X\}$.

A real form $\mathbf{G}$ acts on the complex flag manifold $\mathbf{X}$ by left multiplication, and $\mathbf{X}$ decomposes into a disjoint union of $\mathbf{G}$-orbits. In [21] it is shown that there are finitely many orbits including a unique one which is closed (hence compact).

This orbit M has minimal dimension and is connected. We will refer to M as the *minimal orbit* of $\mathbf{G}$ in $\mathbf{X}$. In particular, the connected component of the identity $\mathbf{G}^\circ$ of $\mathbf{G}$ is transitive on M. Thus, while studying M, we can as well assume that $\mathbf{G} = \mathbf{G}^\circ$ is connected.

Moreover, up to conjugation, we can arrange that the closed orbit is $M = \mathbf{G} \cdot o$, where $o = e\mathbf{Q}$. We shall denote by $\mathbf{G}_+ = \mathbf{G} \cap \mathbf{Q}$ the isotropy subgroup of $\mathbf{G}$ at o and by $\mathfrak{g}_+ = \mathfrak{g} \cap \mathfrak{q}$ its Lie algebra.

The closed orbit M has a natural $\mathbf{G}$-homogeneous *CR* structure induced by its embedding in the complex flag manifold $\mathbf{X}$ and the pair $(\mathfrak{g}, \mathfrak{q})$ is the corresponding *CR* algebra.

6. Parabolic *CR* Algebras

A *CR* algebra $(\mathfrak{g}, \mathfrak{q})$ is *parabolic* if $\mathfrak{g}$ is finite-dimensional and $\mathfrak{q}$ is a parabolic subalgebra of $\hat{\mathfrak{g}}$. We have (see [4, Sec. 5]):

Lemma 6.1. *A parabolic CR algebra* $(\mathfrak{g}, \mathfrak{q})$ *is effective if and only if* $\mathfrak{g}$ *is semisimple and no simple ideal of* $\hat{\mathfrak{g}}$ *is contained in* $\mathfrak{q} \cap \bar{\mathfrak{q}}$.

Proposition 6.2. *Let* $(\mathfrak{g}, \mathfrak{q})$ *be an effective parabolic CR algebra and let* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ *be the decomposition of* $\mathfrak{g}$ *into the direct sum of its simple ideals. Then:*

(1) $\mathfrak{q} = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_\ell$ *where* $\mathfrak{q}_j = \mathfrak{q} \cap \hat{\mathfrak{g}}_j$ *for* $j = 1, \ldots, \ell$;
(2) *for each* $j = 1, \ldots, \ell$, $(\mathfrak{g}_j, \mathfrak{q}_j)$ *is an effective parabolic CR algebra*;
(3) $(\mathfrak{g}, \mathfrak{q})$ *is fundamental* (*resp. ideal, weakly, strictly nondegenerate*) *if and only if for each* $j = 1, \ldots, \ell$, *the CR algebra* $(\mathfrak{g}_j, \mathfrak{q}_j)$ *is fundamental* (*resp. ideal weakly, strictly nondegenerate*).

Thus in the following we can assume, with no loss of generality, that $\mathfrak{g}$ is simple and $\mathfrak{q}$ is parabolic in $\hat{\mathfrak{g}}$.

To an effective parabolic *CR* algebra $(\mathfrak{g}, \mathfrak{q})$ we associate a *CR* manifold $M = M(\mathfrak{g}, \mathfrak{q})$, unique modulo isomorphisms, defined as the orbit $\mathbf{G} \cdot o$ in $\hat{\mathbf{G}}/\mathbf{Q}$, where:

- $\hat{\mathbf{G}}$ is a connected and simply connected Lie group with Lie algebra $\hat{\mathfrak{g}}$;
- $\mathbf{Q} = \mathbf{N}_{\hat{\mathbf{G}}}(\mathfrak{q})$, the normalizer of $\mathfrak{q}$ in $\hat{\mathbf{G}}$, is the parabolic subgroup of $\hat{\mathbf{G}}$ with Lie algebra $\mathfrak{q}$;
- $\mathbf{G}$ is the analytic real subgroup of $\hat{\mathbf{G}}$ with Lie algebra $\mathfrak{g}$.

Note that for a parabolic $(\mathfrak{g}, \mathfrak{q})$ the universal covering $\tilde{M}(\mathfrak{g}, \mathfrak{q})$ of $M(\mathfrak{g}, \mathfrak{q})$ admits a canonical *CR* structure, such that the covering map is a local *CR* isomorphism. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ is semisimple, then:

$$M(\mathfrak{g}, \mathfrak{q}) \simeq M(\mathfrak{g}_1, \mathfrak{q}_1) \times \cdots \times M(\mathfrak{g}_\ell, \mathfrak{q}_\ell) ;$$
$$\tilde{M}(\mathfrak{g}, \mathfrak{q}) \simeq \tilde{M}(\mathfrak{g}_1, \mathfrak{q}_1) \times \cdots \times \tilde{M}(\mathfrak{g}_\ell, \mathfrak{q}_\ell).$$

A real Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is *triangular* if all linear maps $\mathrm{ad}_{\mathfrak{g}}(X) \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g})$ with $X \in \mathfrak{t}$ can be simultaneously represented by triangular matrices in a suitable basis of $\mathfrak{g}$. All maximal triangular subalgebras of $\mathfrak{g}$ are conjugate by inner automorphisms (cf. [16, Sec. 5.4]). A real Lie subalgebra of $\mathfrak{g}$ containing a maximal triangular subalgebra of $\mathfrak{g}$ is called a *t-subalgebra*.

An effective parabolic *CR* algebra $(\mathfrak{g}, \mathfrak{q})$ will be called *minimal* if $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$ is a t-subalgebra of $\mathfrak{g}$.

We observe that a maximal triangular subalgebra of $\mathfrak{g}$ contains a maximal Abelian subalgebra of semisimple elements having real eigenvalues. Hence:

Proposition 6.3. *If* $(\mathfrak{g}, \mathfrak{q})$ *is an effective parabolic minimal CR algebra, then there exists a Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *with maximal vector part contained in* $\mathfrak{q}$.

A Cartan subalgebra with these properties is said to be *adapted* to $(\mathfrak{g}, \mathfrak{q})$.

The *analytic Weyl group* of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is a subgroup of the Weyl group naturally isomorphic to the quotient $\mathbf{N}_{\mathbf{Int}(\mathfrak{g})}(\mathfrak{h})/\mathbf{Z}_{\mathbf{Int}(\mathfrak{g})}(\mathfrak{h})$. Here $\mathbf{Z}$ denotes the centralizer, and $\mathbf{Int}(\mathfrak{g})$ is the group of inner automorphisms of $\mathfrak{g}$.

Theorem 6.4. *Let* $\mathfrak{g}$ *be a semisimple real Lie algebra and* $\mathfrak{q}$ *a parabolic subalgebra of its complexification* $\hat{\mathfrak{g}}$. *Then, up to isomorphisms of CR algebras, there is a unique parabolic minimal CR algebra* $(\mathfrak{g}', \mathfrak{q}')$ *with* $\mathfrak{g}'$ *isomorphic to* $\mathfrak{g}$ *and* $\mathfrak{q}'$ *isomorphic to* $\mathfrak{q}$.

Proof. Fix a maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Its complexification $\hat{\mathfrak{t}}$ is solvable, therefore is contained in a maximal solvable subalgebra, i.e. a Borel subalgebra $\mathfrak{b}$ of $\hat{\mathfrak{g}}$. Modulo an inner automorphism of $\hat{\mathfrak{g}}$, we can assume that $\mathfrak{b} \subset \mathfrak{q}$. The CR algebra $(\mathfrak{g}, \mathfrak{q})$ is parabolic minimal.

Let $\mathfrak{q}$, $\mathfrak{q}'$ be parabolic subalgebras of $\hat{\mathfrak{g}}$ such that $\mathfrak{g}_+ = \mathfrak{q} \cap \mathfrak{g}$ and $\mathfrak{g}'_+ = \mathfrak{q}' \cap \mathfrak{g}$ are t-subalgebras of $\mathfrak{g}$. By an inner automorphism of $\mathfrak{g}$, we can assume that $\mathfrak{g}_+$ and $\mathfrak{g}'_+$ contain the same maximal triangular subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, hence the same maximal Abelian subalgebra of $\mathfrak{g}$ of semisimple elements having real eigenvalues. Then, using another inner automorphism of $\mathfrak{g}$, we can assume that $\mathfrak{q}$ and $\mathfrak{q}'$ contain the same maximal vectorial Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

The inner automorphism of $\hat{\mathfrak{g}}$ transforming $\mathfrak{q}$ into $\mathfrak{q}'$ can now be taken to be an element of the analytic Weyl group, leaving the Cartan subalgebra $\mathfrak{h}$ and hence $\mathfrak{g}$ invariant. It defines a CR isomorphism between $(\mathfrak{g}, \mathfrak{q})$ and $(\mathfrak{g}, \mathfrak{q}')$. □

Effective parabolic minimal *CR* algebras correspond to minimal orbits. In fact we have:

Theorem 6.5. *The CR manifold* $M(\mathfrak{g}, \mathfrak{q})$ *associated to an effective parabolic subalgebra* $(\mathfrak{g}, \mathfrak{q})$ *is compact if and only if* $(\mathfrak{g}, \mathfrak{q})$ *is minimal.*

Proof. Indeed, since $\mathbf{G}$ is a linear group, a $\mathbf{G}$-homogeneous space $\mathbf{G}/\mathbf{G}_+$ is compact if and only if $\mathbf{G}_+$ contains a maximal connected triangular subgroup (see [17, Part II, Chap. 5, Sec. 1.1]), i.e. if $\mathfrak{g}_+$ is a t-subalgebra of $\mathfrak{g}$. □

7. Cross-Marked Satake Diagrams

We recall some facts about Satake diagrams and parabolic subgroups.

Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra with maximal vector part. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that

$$\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^- \quad \text{where } \mathfrak{h}^+ = \mathfrak{h} \cap \mathfrak{k} \text{ and } \mathfrak{h}^- = \mathfrak{h} \cap \mathfrak{p}$$

and set $\mathfrak{h}_{\mathbb{R}} = (i\mathfrak{h}^+) \oplus \mathfrak{h}^-$.

The root system $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ of $\hat{\mathfrak{g}}$ with respect to $\hat{\mathfrak{h}}$ is contained in $\mathfrak{h}^*_{\mathbb{R}}$, the real dual of $\mathfrak{h}_{\mathbb{R}}$. The conjugation σ of $\hat{\mathfrak{g}}$ induces an involution on $\mathfrak{h}^*_{\mathbb{R}}$ (that we still denote by σ), that preserves $\mathcal{R}$.

A root $\alpha \in \mathcal{R}$ is called *real* if $\bar{\alpha} = \sigma(\alpha) = \alpha$, *imaginary* if $\bar{\alpha} = \sigma(\alpha) = -\alpha$. We shall denote by $\mathcal{R}_\bullet$ the set of imaginary roots in $\mathcal{R}$.

For a Weyl chamber C, denote by $\mathcal{R}^+(C)$ the set of positive roots. If C' is another Weyl chamber, let $w_{C,C'}$ be the unique element of the Weyl group sending C to C'. We have (see [5]):

Proposition 7.1. *There exists a Weyl chamber* C, *unique modulo the action of the analytic Weyl group, such that* $\bar{\alpha} \succ 0$ *for all* $\alpha \in \mathcal{R}^+(C) \setminus \mathcal{R}_\bullet$.

This Weyl chamber determines a basis $\mathcal{B} = \mathcal{B}(C)$ of $\mathcal{R}$. The map

$$\epsilon_C = \sigma \circ w_{C,\bar{C}} \tag{7.1}$$

is an involution of $\mathcal{R}$ that preserves C, and thus $\mathcal{B}$. Moreover, for every root $\alpha \in \mathcal{B} \setminus \mathcal{R}_\bullet$, there are integers $n_{\alpha,\beta} \geq 0$ such that

$$\bar{\alpha} = \varepsilon_C(\alpha) + \sum_{\beta \in \mathcal{B} \cap \mathcal{R}_\bullet} n_{\alpha,\beta}\beta\,. \tag{7.2}$$

The *Satake diagram* $\mathcal{S}$ of $\mathfrak{g}$ is obtained by the Dynkin diagram of $\hat{\mathfrak{g}}$ whose nodes correspond to the roots in $\mathcal{B}$ by painting black those corresponding to imaginary roots and joining by a curved arrow those corresponding to pairs of distinct nonimaginary roots α, β with $\epsilon_C(\alpha) = \beta$.

There is a bijective correspondence between real semisimple Lie algebras and Satake diagrams.

Let $\hat{\mathfrak{g}}$ be a complex semisimple Lie algebra, $\hat{\mathfrak{h}}$ any Cartan subalgebra, $\mathcal{R}$ the root system, C a Weyl chamber, $\mathcal{R}^+$ and $\mathcal{R}^-$ the corresponding sets of positive and negative roots, $\mathcal{B} = \{\alpha_1, \ldots, \alpha_\ell\}$ the corresponding basis.

Denote by $\hat{\mathfrak{g}}^\alpha$ the eigenspace in $\hat{\mathfrak{g}}$ for the root α. If $\beta = \sum_i n_i\alpha_i$, $n_i \geq 0$, define the *support* of β: $\mathrm{supp}(\beta) = \{\alpha_i \mid n_i > 0\}$.

Let Φ be a subset of $\mathcal{B}$. The set $\check{\Phi}^r$ of those $\beta \in \mathcal{R}^-$ for which $\mathrm{supp}(\beta) \cap \Phi = \emptyset$ is a closed system of roots (i.e. $\beta_1, \beta_2 \in \check{\Phi}^r$ and $\beta_1 + \beta_2 \in \mathcal{R} \Rightarrow \beta_1 + \beta_2 \in \check{\Phi}^r$). Then

$$\mathfrak{q}_\Phi = \hat{\mathfrak{h}} \oplus \sum_{\beta \in \mathcal{R}^+} \hat{\mathfrak{g}}^\beta \oplus \sum_{\beta \in \check{\Phi}^r} \hat{\mathfrak{g}}^\beta \tag{7.3}$$

is a parabolic subalgebra of $\hat{\mathfrak{g}}$, and every parabolic subalgebra of $\hat{\mathfrak{g}}$ can be described in this way. More precisely:

Proposition 7.2. *Let $\hat{\mathfrak{g}}$ be a complex semisimple Lie algebra and $\mathfrak{q}$ a parabolic subalgebra. Then there exist a Cartan subalgebra $\hat{\mathfrak{h}}$, a Weyl chamber C and a subset $\Phi \subset \mathcal{B}$ such that $\mathfrak{q} = \mathfrak{q}_\Phi$.*

To a parabolic subalgebra $\mathfrak{q}_\Phi$ we associate a *cross-marked Dynkin diagram*, consisting of the Dynkin diagram of $\mathfrak{g}$ with cross-marks on the nodes corresponding to roots in Φ. The correspondence between isomorphism classes of parabolic subalgebras and cross-marked Dynkin diagrams is bijective.

If $(\mathfrak{g}, \mathfrak{q})$ is an effective parabolic minimal *CR* algebra, these two constructions are compatible, in the following sense (see [4, Sec. 6]):

Theorem 7.3. *Let $(\mathfrak{g}, \mathfrak{q})$ be an effective parabolic minimal CR algebra. Then there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with maximal vector part, a Weyl chamber C for $\mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$ and a subset Φ of $\mathcal{B}(C)$ such that $\mathfrak{q} = \mathfrak{q}_\Phi$.*

Then to an effective parabolic minimal *CR* algebra we can associate a *cross-marked Satake diagrams* $(\mathcal{S}, \Phi)$ where $\mathcal{S}$ is the Satake diagram of $\mathfrak{g}$ and $\mathfrak{q} = \mathfrak{q}_\Phi$; this correspondence is bijective, modulo isomorphisms.

We set up some notations that we will use in the following sections. If $\mathfrak{q} = \mathfrak{q}_\Phi$, let $\mathcal{Q} = \mathcal{R}^+ \cup \check{\Phi}^r$, $\mathcal{Q}^r = \check{\Phi}^r \cup (-\check{\Phi}^r)$ and $\mathcal{Q}^n = \mathcal{Q} \setminus \mathcal{Q}^r$. To the partition $\mathcal{Q} = \mathcal{Q}^r \cup \mathcal{Q}^n$ corresponds a direct sum decomposition $\mathfrak{q} = \mathfrak{q}^r \oplus \mathfrak{q}^n$, where

$$\mathfrak{q}^r = \hat{\mathfrak{h}} \oplus \sum_{\beta \in \mathcal{Q}^r} \hat{\mathfrak{g}}^\beta \quad \text{and} \quad \mathfrak{q}^n = \sum_{\beta \in \mathcal{Q}^n} \hat{\mathfrak{g}}^\beta$$

are the *reductive* and *nilpotent* part of $\mathfrak{q}$.

8. *CR* Fibrations for Parabolic Minimal *CR* Algebras

In this section we discuss morphisms of CR algebras of the special form $(\mathfrak{g}, \mathfrak{q}) \to (\mathfrak{g}, \mathfrak{q}')$, for $\mathfrak{q} \subset \mathfrak{q}'$. They have been called in [14] $\mathfrak{g}$-*equivariant fibrations* and describe at the level of CR algebras the corresponding $\mathbf{G}$-equivariant smooth fibrations $M(\mathfrak{g}, \mathfrak{q}) \to M(\mathfrak{g}, \mathfrak{q}')$.

We keep the notation of the previous sections. In particular, $\mathfrak{g}$ is a semisimple real Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ with maximal vector part, $\mathcal{R} = \mathcal{R}(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})$, C a Weyl chamber adapted to the conjugation σ in $\mathfrak{h}^*_{\mathbb{R}}$ induced by the real form $\mathfrak{g}$ of $\hat{\mathfrak{g}}$ (as in Proposition 7.1), $\mathcal{B} = \mathcal{B}(C)$ is the set of simple roots in $\mathcal{R}^+ = \mathcal{R}^+(C)$.

Let $\Psi \subset \Phi \subset \mathcal{B}$. Then $\mathfrak{q}_\Phi \subset \mathfrak{q}_\Psi$ and the identity on $\mathfrak{g}$ defines a natural $\mathfrak{g}$-equivariant morphism of CR algebras:

$$\pi : (\mathfrak{g}, \mathfrak{q}_\Phi) \to (\mathfrak{g}, \mathfrak{q}_\Psi)\,. \tag{8.1}$$

Its fiber is the CR algebra $(\mathfrak{g}', \mathfrak{q}')$ where:

$$\begin{cases} \mathfrak{g}' = \mathfrak{g} \cap \mathfrak{q}_\Psi = \mathfrak{g} \cap \bar{\mathfrak{q}}_\Psi\,, \quad \hat{\mathfrak{g}}' = \mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Psi, \\ \mathfrak{q}' = \hat{\mathfrak{q}}_\Phi \cap \hat{\mathfrak{g}}' = \hat{\mathfrak{q}}_\Phi \cap \mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Psi = \mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Psi\,. \end{cases} \tag{8.2}$$

Denote by $\mathcal{R}'$ and $\mathcal{Q}'$ the sets of roots $\alpha \in \mathcal{R}$ for which $\hat{\mathfrak{g}}^\alpha$ is contained in $\hat{\mathfrak{g}}'$ and $\hat{\mathfrak{q}}'$, respectively:

$$\begin{cases} \mathcal{R}' = \mathcal{Q}_\Psi \cap \bar{\mathcal{Q}}_\Psi, \\ \mathcal{Q}' = \mathcal{Q}_\Phi \cap \bar{\mathcal{Q}}_\Psi\,. \end{cases} \tag{8.3}$$

The CR algebra $(\mathfrak{g}', \mathfrak{q}')$ in general is neither parabolic nor effective. However its effective quotient *is* parabolic minimal. Define:

$$\begin{cases} \mathcal{R}'' = \mathcal{R}' \cap (-\mathcal{R}') \ = \ \mathcal{Q}^r_\Psi \cap \bar{\mathcal{Q}}^r_\Psi, \\ \mathcal{Q}'' = \mathcal{Q}' \cap \mathcal{R}'' \end{cases} \tag{8.4}$$

and set:

$$\begin{cases} \hat{\mathfrak{g}}'' = \hat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \mathcal{R}''} \hat{\mathfrak{g}}^\alpha, \\ \mathfrak{q}'' = \mathfrak{q}' \cap \hat{\mathfrak{g}}''. \end{cases} \tag{8.5}$$

Then $\mathcal{R}''$ is σ-invariant, $\hat{\mathfrak{g}}'' = \mathfrak{q}^r_\Psi \cap \bar{\mathfrak{q}}^r_\Psi$ is reductive and $\mathfrak{q}''$ is parabolic in $\hat{\mathfrak{g}}''$. Furthermore $\mathcal{B}'' = \mathcal{B} \cap \mathcal{R}''$ is a basis of $\mathcal{R}''$.

Hence we obtain (see [4, Sec. 7]):

Proposition 8.1. *The CR algebra* $(\mathfrak{g}'', \mathfrak{q}'')$ *is parabolic minimal. Its cross-marked Satake diagram* $(\mathcal{S}'', \Phi'')$ *is the subdiagram of* $(\mathcal{S}, \Phi)$ *consisting of the simple roots* α *such that either one of the following conditions holds*:

(1) $\alpha \in \mathcal{R}_\bullet \setminus \Psi$;
(2) $\alpha \notin \mathcal{R}_\bullet$ *and* $(\{\alpha\} \cup \operatorname{supp}(\bar{\alpha})) \cap \Psi = \emptyset$.

The cross-marks are left on the nodes corresponding to roots in $\Phi \cap \mathcal{B}''$.

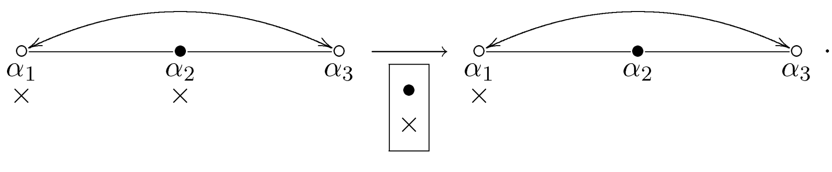
We say that a Satake diagram is *σ-connected* if either it is connected or consists of two connected components, joined by curved arrows.

Theorem 8.2. *Let* (8.1) *be a* $\mathfrak{g}$*-equivariant fibration. Then the effective quotient of its fiber is the parabolic minimal CR algebra whose cross-marked Satake diagram consists of the union of all* σ*-connected components of the diagram* $\mathcal{S}''$ *described in Proposition* 8.1, *containing at least one cross-marked node.*

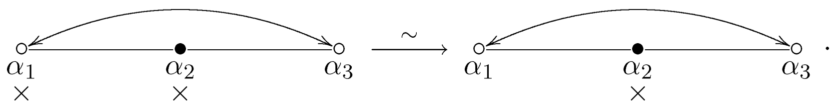
Recall that a $\mathfrak{g}$-equivariant morphism of CR algebras (8.1) is a CR -fibration if the quotient map $\mathfrak{q}_\Phi/(\mathfrak{q}_\Phi \cap \bar{\mathfrak{q}}_\Phi) \to \mathfrak{q}_\Psi/(\mathfrak{q}_\Psi \cap \bar{\mathfrak{q}}_\Psi)$ is onto.

The condition that (8.1) is a *CR* fibration is equivalent to the fact that the quotient map $M(\mathfrak{g}, \mathfrak{q}_\Phi) \to M(\mathfrak{g}, \mathfrak{q}_\Psi)$ is a *CR* fibration with fiber $M(\mathfrak{g}'', \mathfrak{q}'')$. This condition is always satisfied if $(\mathfrak{g}, \mathfrak{q}_\Psi)$ is totally real, indeed in this case $\mathfrak{q}_\Psi/(\mathfrak{q}_\Psi \cap \overline{\mathfrak{q}}_\Psi) = 0$.

Example 8.3. Let $\mathfrak{g} = \mathfrak{su}(1,3)$ and let $\Phi = \{\alpha_1, \alpha_2\}$, $\Psi = \{\alpha_1\}$. Then the cross-marked Satake diagrams corresponding to the CR algebra $(\mathfrak{g}, \mathfrak{q}_\Phi)$, the basis $(\mathfrak{g}, \mathfrak{q}_\Psi)$ and the corresponding effective fiber are given by:



In the case $\Psi = \{\alpha_2\}$ we have instead:



The fiber is trivial and the map is a CR morphism, but not a CR isomorphism. The corresponding map $M(\mathfrak{g}, \mathfrak{q}_\Phi) \to M(\mathfrak{g}, \mathfrak{q}_\Psi)$ is an analytic diffeomorphism and a CR map, but not a CR diffeomorphism.

9. The Fundamental Reduction for Parabolic Minimal *CR* Algebras

We give a criterion to read off the property of being fundamental from the cross-marked Satake diagram:

Proposition 9.1. *An effective parabolic minimal CR algebra* $(\mathfrak{g}, \mathfrak{q}_\Phi)$ *is fundamental if and only if its corresponding cross-marked Satake diagram* $(\mathcal{S}, \Phi)$ *has the property*:

$$\alpha \in \Phi \backslash \mathcal{R}_\bullet \Rightarrow \epsilon_C(\alpha) \notin \Phi\,.$$

Here ϵ_C is the involution in $\mathcal{B}(C)$ defined in (7.1).

Proof. Assume that α_1 and $\alpha_2 = \epsilon_C(\alpha_1)$ both belong to Φ, and let $\Psi = \{\alpha_1, \alpha_2\}$. Then $\Psi \subset \Phi$ and hence $\mathfrak{q}_\Phi \subset \mathfrak{q}_\Psi$. To show that $(\mathfrak{g}, \mathfrak{q}_\Phi)$ is not fundamental, it is sufficient to check that $\mathfrak{q}_\Psi = \bar{\mathfrak{q}}_\Psi$. To this aim it suffices to verify that $\mathcal{Q}^n_\Psi = \overline{\mathcal{Q}}^n_\Psi$. Let $\mathcal{B}(C) = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$. Every root $\alpha \in \mathcal{Q}^n_\Psi$ can be written in the form $\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ with $k_1 + k_2 > 0$. Since C is adapted to the conjugation σ, using (7.2) we obtain:

$$\bar{\alpha} = \sum_{i=1}^{\ell} k_i \epsilon_C(\alpha_i) + \sum_{\beta \in \mathcal{B} \cap \mathcal{R}_\bullet} k_{\alpha,\beta} \beta = \sum_{i=1}^{\ell} k'_i \alpha_i,$$

with $k'_1 + k'_2 = k_2 + k_1 > 0$, showing that also $\bar{\alpha} \in \mathcal{Q}^n_\Psi$. This shows that the condition is necessary.

Assume *vice versa* that there exists a proper parabolic subalgebra $\mathfrak{q}'$ of $\hat{\mathfrak{g}}$ with $\mathfrak{q}_\Phi \subset \mathfrak{q}' = \bar{\mathfrak{q}}'$. Then $\mathfrak{q}' = \mathfrak{q}_\Psi$ for some $\Psi \subset \Phi$, $\Psi \neq \emptyset$. Since $\overline{\mathcal{Q}}^n_\Psi = \mathcal{Q}^n_\Psi \subset \mathcal{R}^+(C)$, we have $\Psi \cap \mathcal{R}_\bullet = \emptyset$. Hence, again by (7.2), we obtain that $\epsilon_C(\alpha) \in \Psi$ for all $\alpha \in \Psi$. □

From Propositions 9.1 and 8.1 we obtain:

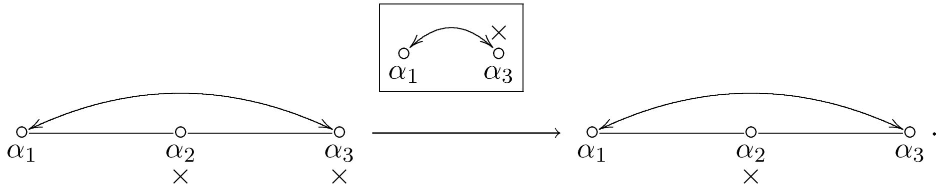
Theorem 9.2. *Let* $(\mathfrak{g}, \mathfrak{q}_\Phi)$ *be an effective parabolic minimal CR algebra and let* $(\mathcal{S}, \Phi)$ *be its corresponding cross-marked Satake diagram. Let*

$$\Psi = \{\alpha \in \Phi \backslash \mathcal{R}_\bullet \,|\, \epsilon_C(\alpha) \in \Phi\}.$$

Then

(1) *The diagram* $\mathcal{S}'$ *obtained from* $\mathcal{S}$ *by erasing all the nodes corresponding to the roots in* Ψ *and the lines and arrows issued from them is still a Satake diagram, corresponding to a semisimple real Lie algebra* $\mathfrak{g}'$.
(2) $(\mathfrak{g}, \mathfrak{q}_\Psi)$ *is a totally real effective parabolic minimal CR algebra.*
(3) *The natural map* $(\mathfrak{g}, \mathfrak{q}_\Phi) \to (\mathfrak{g}, \mathfrak{q}_\Psi)$, *defined by the inclusion* $\mathfrak{q}_\Phi \subset \mathfrak{q}_\Psi$, *is a* $\mathfrak{g}$-*equivariant CR fibration. The effective quotient of its fiber is the fundamental parabolic minimal CR algebra* $(\mathfrak{g}'', \mathfrak{q}_{\Phi'})$, *associated to the cross-marked Satake diagram* $(\mathcal{S}'', \Phi')$, *where* $\Phi' = \Phi \backslash \Psi$ *and* $\mathcal{S}''$ *is the union of the* σ-*connected components of* $\mathcal{S}'$ *that contain some root of* Φ'.

Example 9.3. Let $\mathfrak{g} \simeq \mathfrak{su}(2,2)$ and let $\Phi = \{\alpha_2, \alpha_3\}$ (we refer to the diagram given). We have $\epsilon_C(\alpha_i) = \alpha_{4-i}$ for $i = 1, 2, 3$ and hence $\Psi = \{\alpha \in \Phi \,|\, \epsilon_C(\alpha) \in \Phi\} = \{\alpha_2\}$. In particular, $(\mathfrak{g}, \mathfrak{q}_{\{\alpha_2,\alpha_3\}})$ is not fundamental. We obtain a $\mathfrak{g}$-equivariant CR fibration $(\mathfrak{g}, \mathfrak{q}_{\{\alpha_2,\alpha_3\}}) \to (\mathfrak{g}, \mathfrak{q}_{\{\alpha_2\}})$ with fundamental fiber $(\mathfrak{g}', \mathfrak{q}'_{\{\alpha_3\}})$, with $\mathfrak{g}' \simeq \mathfrak{sl}(2, \mathbb{C})$.



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